ตัวอย่างการเขียนแบบจำลอง
ในรูปเมตริกซ์

จากแบบจำลองที่เคยศึกษามาใน ST 204

ถ้าแบบจำลองในรูปเมตริกซ์คือ $Y = X\beta + e$

(1) จงเขียนเมตริกซ์ Y, X, β, e, $X'X$, $X'Y$, $Y'Y$, $e'e$

(2) ถ้า Normal equations คือ $X'X\hat{\beta} = X'Y$

จงเขียน Normal equations ซึ่งไม่อยู่ในรูปเมตริกซ์

ตัวอย่างที่ 1 แบบจำลองการถดถอยอย่างง่าย (Simple regression model):

$y_i = \beta_0 + \beta_1 x_i + e_i,\ i = 1,\dots,n$

ค่าสังเกต (Observations): (x_i, y_i), $i = 1,\dots,n$

$$
\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + e_1 \\
y_2 &= \beta_0 + \beta_1 x_2 + e_2 \\
&\cdots \\
y_n &= \beta_0 + \beta_1 x_n + e_n
\end{aligned}
$$

แบบจำลองในรูปเมตริกซ์คือ $Y = X\beta + e$

โดยที่

$$
Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1},\quad
X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdots & \\ 1 & x_n \end{bmatrix}_{n\times 2},\quad
\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix},\quad
e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}_{n\times 1}
$$

$$
X'X = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix}
\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdots & \\ 1 & x_n \end{bmatrix}
= \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}_{2\times 2}
$$

$$X'Y = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}_{2\times1}$$

$$Y'Y = \Sigma y_i^2 \text{ , } e'e = \Sigma e_i^2 \quad \text{โดยที่ } \Sigma \text{ คือ } \sum_{i=1}^{n}$$

$$(X'X)^{-1} = \{1/[n\Sigma x_i^2 - (\Sigma x_i)^2]\} \begin{bmatrix} \Sigma x_i^2 & \text{[illegible]} \\ -\Sigma x_i & n \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\Sigma x_i^2}{n\Sigma x_i^2 - (\Sigma x_i)^2} & \dfrac{-\Sigma x_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} \\ \dfrac{-\Sigma x_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} & \dfrac{n}{n\Sigma x_i^2 - (\Sigma x_i)^2} \end{bmatrix}_{2\times2}$$

(2) Normal equations: $X'X\hat{\beta} = X'Y$

$$\text{หรือ} \quad \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_o \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

หรือ $\quad n\hat{\beta}_o + \hat{\beta}_1 \Sigma x_i = \Sigma y_i \qquad \dots(1)$

$\hat{\beta}_o \Sigma x_i + \hat{\beta}_1 \Sigma x_i^2 = \Sigma x_i y_i \qquad \dots(2)$

(3) ถ้ากำหนด : $n = 10$, $\Sigma y_i = 1100$, $\Sigma x_i = 500$, $\Sigma x_i^2 = 28400$,

$\Sigma x_i y_i = 61800$

จงเขียน $X'X$, $X'Y$, $(X'X)^{-1}$, $(X'X)^{-1}X'Y$

$$x'x = \begin{bmatrix} 10 & 500 \\ 500 & 28400 \end{bmatrix}$$

$$X'Y = \begin{bmatrix} 1100 \\ 61800 \end{bmatrix}$$

$$(X'X)^{-1} = \{1/[10(28400)-(500)^2]\} \begin{bmatrix} 28400 & -500 \\ -500 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0.83529412 & -0.01470588 \\ -0.01470588 & 0.00029412 \end{bmatrix}$$

$$(X'X)^{-1}X'Y = (1/34000) \begin{bmatrix} 28400 & -500 \\ -500 & 10 \end{bmatrix} \begin{bmatrix} 1100 \\ 61800 \end{bmatrix}$$

$$= (1/34000) \begin{bmatrix} 340000 \\ 68000 \end{bmatrix} = \begin{bmatrix} 10.0 \\ 2.0 \end{bmatrix}$$ ซึ่งคือ $\hat{\beta}$

ดังนั้น $\hat{\beta}_o = 10.0$ และ $\hat{\beta}_1 = 2.0$

ตัวอย่างที่ 2 แบบจำลองการถดถอยเชิงพหุ (Multiple regression model)

มีตัวแปรอิสระ 3 ตัว

$$y_i = \beta_o + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + e_i \; , \; i = 1,\ldots,n$$

ค่าสังเกต: $(x_{i1}, x_{i2}, x_{i3}, y_i)$, $i = 1,\ldots, n$

$$y_1 = \beta_0 + \beta_1 X_{11} + \beta_2 X_{12} + \beta_3 X_{13} + e_1$$
$$y_2 = \beta_0 + \beta_1 X_{21} + \beta_2 X_{22} + \beta_3 X_{23} + e_2$$
$$\cdots$$
$$y_n = \beta_0 + \beta_1 X_{n1} + \beta_2 X_{n2} + \beta_3 X_{n3} + e_n$$

(1) แบบจำลองในรูปเมตริกซ์คือ $Y = X\beta + e$

โดยที่

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times1}, \quad X = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{13} \\ 1 & X_{21} & X_{22} & X_{23} \\ & & \cdots & \\ 1 & X_{n1} & X_{n2} & X_{n3} \end{bmatrix}_{n\times4},$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}_{4\times1}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}_{n\times1}$$

$$X'X = \begin{bmatrix} n & \Sigma X_{i1} & \Sigma X_{i2} & \Sigma X_{i3} \\ \Sigma X_{i1} & \Sigma X^2_{i1} & \Sigma X_{i1}X_{i2} & \Sigma X_{i1}X_{i3} \\ \Sigma X_{i2} & \Sigma X_{i1}X_{i2} & \Sigma X^2_{i2} & \Sigma X_{i2}X_{i3} \\ \Sigma X_{i3} & \Sigma X_{i1}X_{i3} & \Sigma X_{i2}X_{i3} & \Sigma X^2_{i3} \end{bmatrix}_{4\times4}$$

$$X'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma X_{i1}y_i \\ \Sigma X_{i2}y_i \\ \Sigma X_{i3}y_i \end{bmatrix}_{4\times1}$$

$$Y'Y = \Sigma y_i^2 \;,\; e'e = \Sigma e_i^2 \quad \text{โดยที่ } \Sigma \text{ คือ } \sum_{i=1}^{n}$$

(2)

$$\begin{aligned}
\hat{\beta}_0 n &+ \hat{\beta}_1 \Sigma x_{i1} &+ \hat{\beta}_2 \Sigma x_{i2} &+ \hat{\beta}_3 \Sigma x_{i3} &= \Sigma y_i \\
\hat{\beta}_0 \Sigma x_{i1} &+ \hat{\beta}_1 \Sigma x^2_{i1} &+ \hat{\beta}_2 \Sigma x_{i1}x_{i2} &+ \hat{\beta}_3 \Sigma x_{i1}x_{i3} &= \Sigma x_{i1}y_i \\
\hat{\beta}_0 \Sigma x_{i2} &+ \hat{\beta}_1 \Sigma x_{i1}x_{i2} &+ \hat{\beta}_2 \Sigma x^2_{i2} &+ \hat{\beta}_3 \Sigma x_{i2}x_{i3} &= \Sigma x_{i2}y_i \\
\hat{\beta}_0 \Sigma x_{i3} &+ \hat{\beta}_1 \Sigma x_{i1}x_{i3} &+ \hat{\beta}_2 \Sigma x_{i2}x_{i3} &+ \hat{\beta}_3 \Sigma x^2_{i3} &= \Sigma x_{i3}y_i
\end{aligned}$$

ตัวอย่างที่ 3 แบบจำลองการถดถอยแบบโพลิโนเมียล [Polynomial regression (4th degree) model]:

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \beta_3 x_i^3 + \beta_4 x_i^4 + e_i, \; i=1,\ldots,n$$

ค่าสังเกต: (x_i, y_i), $i = 1,\ldots, n$

$$\begin{aligned}
y_1 &= \beta_0 + \beta_1 x_1 + \beta_2 x_1^2 + \beta_3 x_1^3 + \beta_4 x_1^4 + e_1 \\
y_2 &= \beta_0 + \beta_1 x_2 + \beta_2 x_2^2 + \beta_3 x_2^3 + \beta_4 x_2^4 + e_2 \\
&\cdots \\
y_n &= \beta_0 + \beta_1 x_n + \beta_2 x_n^2 + \beta_3 x_n^3 + \beta_4 x_n^4 + e_n
\end{aligned}$$

(1) แบบจำลองในรูปเมตริกซ์คือ $Y = X\beta + e$

โดยที่

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times1}, \quad X = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 \\ & & \cdots & & \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 \end{bmatrix}_{n\times4},$$

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}_{5\times1}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}_{n\times1}$$

$$X'X = \begin{bmatrix} n & \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 \\ \Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 & \Sigma x_i^5 \\ \Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 & \Sigma x_i^5 & \Sigma x_i^6 \\ \Sigma x_i^3 & \Sigma x_i^4 & \Sigma x_i^5 & \Sigma x_i^6 & \Sigma x_i^7 \\ \Sigma x_i^4 & \Sigma x_i^5 & \Sigma x_i^6 & \Sigma x_i^7 & \Sigma x_i^8 \end{bmatrix}_{5\times 5}$$

$$X'Y = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \\ \Sigma x_i^3 y_i \\ \Sigma x_i^4 y_i \end{bmatrix}_{5\times 1}$$

$Y'Y = \Sigma y_i^2$, $e'e = \Sigma e_i^2$ โดยที่ Σ คือ $\sum_{i=1}^{n}$

(2)
$$\begin{aligned}
\hat{\beta}_0 n &+ \hat{\beta}_1 \Sigma x_i + \hat{\beta}_2 \Sigma x_i^2 + \hat{\beta}_3 \Sigma x_i^3 + \hat{\beta}_3 \Sigma x_i^4 = \Sigma y_i \\
\hat{\beta}_0 \Sigma x_i &+ \hat{\beta}_1 \Sigma x_i^2 + \hat{\beta}_2 \Sigma x_i^3 + \hat{\beta}_3 \Sigma x_i^4 + \hat{\beta}_3 \Sigma x_i^5 = \Sigma x_i y_i \\
\hat{\beta}_0 \Sigma x_i^2 &+ \hat{\beta}_1 \Sigma x_i^3 + \hat{\beta}_2 \Sigma x_i^4 + \hat{\beta}_3 \Sigma x_i^5 + \hat{\beta}_3 \Sigma x_i^6 = \Sigma x_i^2 y_i \\
\hat{\beta}_0 \Sigma x_i^3 &+ \hat{\beta}_1 \Sigma x_i^4 + \hat{\beta}_2 \Sigma x_i^5 + \hat{\beta}_3 \Sigma x_i^6 + \hat{\beta}_3 \Sigma x_i^7 = \Sigma x_i^3 y_i \\
\hat{\beta}_0 \Sigma x_i^4 &+ \hat{\beta}_1 \Sigma x_i^5 + \hat{\beta}_2 \Sigma x_i^6 + \hat{\beta}_3 \Sigma x_i^7 + \hat{\beta}_3 \Sigma x_i^8 = \Sigma x_i^4 y_i
\end{aligned}$$

<u>ตัวอย่างที่ 4</u> แบบจำลองของการวิเคราะห์ความแปรปรวนสำหรับการจำแนกทางเดียว (One-way Analysis of Variance model):

$$Y_{ij} = \mu_i + e_{ij} \text{ , } i = 1,\ldots, t; \; j = 1,\ldots, n_i$$

ค่าสังเกต:

	วิธีการ			
	1	2 ...	t	
	y_{11}	y_{21}	y_{t1}	
	y_{12}	y_{22}	y_{t2}	
	:	:	:	
	y_{1n_1}	y_{2n_2}	y_{tn_t}	
รวม	$y_{1.}$	$y_{2.}$	$y_{t.}$	$y_{..}$
ขนาดตัวอย่าง	n_1	n_2	n_t	$n = \sum_{i=1}^{t} n_i$

โดยที่ $y_{i.} = \sum_{j=1}^{n} y_{ij}$

$$y_{..} = \sum_{i=1}^{t} \sum_{j=1}^{n} y_{ij}$$

$$Y_{11} = \mu_1 + e_{11}$$

...

$$Y_{1n_1} = \mu_1 + e_{1n_1}$$

$$Y_{21} = \mu_2 + e_{21}$$

...

$$Y_{2n_2} = \mu_2 + e_{2n_2}$$

...

$$Y_{t1} = \mu_t + e_{t1}$$

...

$$y_{tn_t} = \mu_t + e_{tn_t}$$

(1) โดยที่

$$Y = \begin{bmatrix} y_{11} \\ \vdots \\ y_{1n_1} \\ \hline y_{21} \\ \vdots \\ y_{2n_2} \\ \hline \vdots \\ \hline y_{t1} \\ \vdots \\ y_{tn_t} \end{bmatrix}_{n\times 1}, \quad X = \begin{bmatrix} 1 & 0 & \dots & 0 \\ & \dots & & \\ 1 & 0 & \dots & 0 \\ \hline 0 & 1 & \dots & 0 \\ & \dots & & \\ 0 & 1 & \dots & 0 \\ \hline & \dots & & \\ \hline 0 & 0 & \dots & 1 \\ & \dots & & \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n\times t}, \quad e = \begin{bmatrix} e_{11} \\ \\ e_{1n_1} \\ \hline e_{21} \\ \\ e_{2n_2} \\ \hline \dots \\ \hline e_{t1} \\ \dots \\ e_{tn_t} \end{bmatrix}_{n\times 1}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_t \end{bmatrix}_{t\times 1}$$

$$X'X = \begin{bmatrix} n_1 & 0 & 0 & \dots & 0 \\ 0 & n_2 & 0 & \dots & 0 \\ & & \dots & & \\ 0 & 0 & 0 & \dots & n_t \end{bmatrix}_{t\times t}, \quad X'Y = \begin{bmatrix} y_{1.} \\ y_{2.} \\ \vdots \\ y_{t.} \end{bmatrix}_{t\times 1}$$

$$Y'Y = \sum_{i=1}^{t} \sum_{j=1}^{n} y_{ij}^2, \quad e'e = \sum_{i=1}^{t} \sum_{j=1}^{n} e_{ij}^2$$