(iii) The probability of no heads (i.e. all failures) is $q^6 = (\frac{1}{2})^6 = \frac{1}{64}$, and so the probability of at least one head is $1 - q^6 = 1 - \frac{1}{64} = \frac{63}{64}$.

Example 6.2: A fair die is tossed 7 times; call a toss a success if a 5 or a 6 appears. Then $n = 7$, $p = P(\{5, 6\}) = \frac{1}{3}$ and $q = 1 - p = \frac{2}{3}$.

(i) The probability that 5 or a 6 occurs exactly 3 times (i.e. $k = 3$) is

$$b(3; 7, \tfrac{1}{3}) = \binom{7}{3}(\tfrac{1}{3})^3(\tfrac{2}{3})^4 = \frac{560}{2187}$$

(ii) The probability that a 5 or a 6 never occurs (i.e. all failures) is $q^7 = (\frac{2}{3})^7 = \frac{128}{2187}$; hence the probability that a 5 or a 6 occurs at least once is $1 - q^7 = \frac{2059}{2187}$.

If we regard n and p as constant, then the above function $P(k) = b(k; n, p)$ is a discrete probability distribution:

k	0	1	2	...	n
$P(k)$	q^n	$\binom{n}{1} q^{n-1}p$	$\binom{n}{2} q^{n-2}p^2$	...	p^n

It is called the *binomial distribution* since for k = 0, 1, 2, ..., n it corresponds to the successive terms of the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1}p + \binom{n}{2} q^{n-2}p^2 + \ldots + p^n$$

This distribution is also called the Bernoulli distribution, and independent trials with two outcomes are called Bernoulli trials.

Properties of this distribution follow:

Theorem 6.2:

Binomial distribution	
Mean	$\mu = np$
Variance	$\sigma^2 = npq$
Standard deviation	$\sigma = \sqrt{npq}$

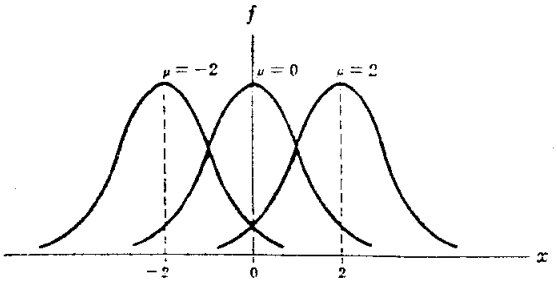
Example 6.3: A fair die is tossed 180 times. The expected number of sixes is $\mu = np = 180 \cdot \frac{1}{6} = 30$. The standard deviation is $\sigma = \sqrt{npq} = \sqrt{180 \cdot \frac{1}{6} \cdot \frac{5}{6}} = 5$.

NORMAL DISTRIBUTION

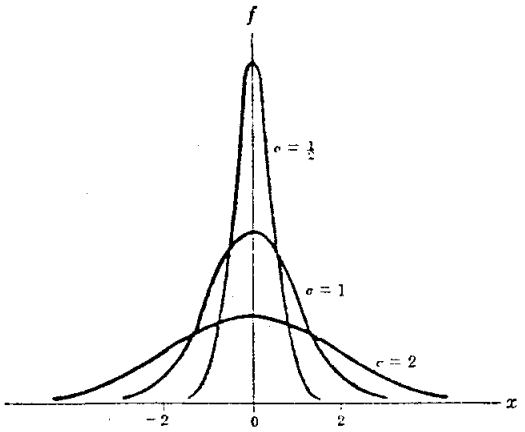
The *normal* (or : *Gaussian) distribution* or *curve* is defined as follows :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

Where μ and $\sigma > 0$ are arbitrary constants. This function is certainly one of the most important examples of a continuous probability distribution. The two diagrams below show the changes in f as μ varies and as σ varies. In particular, observe that these bellshaped curves are symmetric about $x = \mu$.



Normal distributions with σ fixed ($\sigma = 1$)



Normal distributions with μ fixed ($\mu = 0$)

Properties of the normal distribution follow :

Theorem 6.3 :

Normal distribution	
Mean	μ
Variance	σ^2
Standard deviation	σ

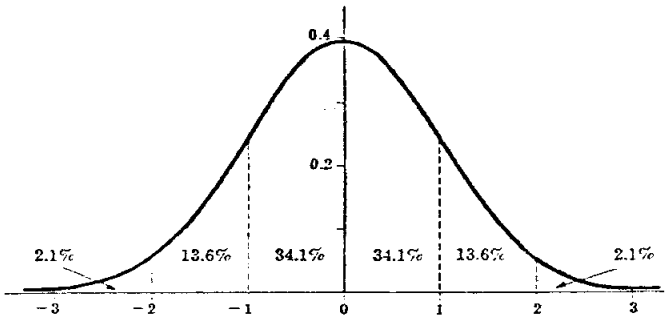
We denote the above normal distribution with mean μ and variance σ^2 by

$$N(\mu, \sigma^2)$$

If we make the substitution $t = (x-\mu)/\sigma$ in the above formula for $N(\mu, \sigma^2)$ we obtain the *standard normal distribution* or *curve*

$$\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-1/2\,t^2}$$

which has mean $\mu = 0$ and variance $\sigma^2 = 1$. The graph of this distribution appears below. We note that for $-1 \leqslant t \leqslant 1$ we obtain 68.2% of the area under the curve, and for $-2 \leqslant t \leqslant 2$ we obtain 95.4% of the area under the curve.



Normal distribution $N(0, 1)$

Table SNCA gives the area under the standard normal curve between t = 0 and any positive value of t. The symmetry of the curve about t = 0 permits us to obtain the area between any two values of t.

Now let X be a continuous random variable with a normal distribution; we frequently say that X is *normally distributed.* We compute the probability that X lies between a and b, denoted by $P(a \leqslant X \leqslant b)$, as follows. First we change a and b into *standard units*

$$a' = (a-\mu)\sigma \quad \text{and} \quad b' = (b-\mu)\sigma$$

respectively Then

$$P(a \leqslant X \leqslant b) = P(a' \leqslant X^* \leqslant b')$$
$$= \text{area under the standard normal curve between } a' \text{ and } b'$$

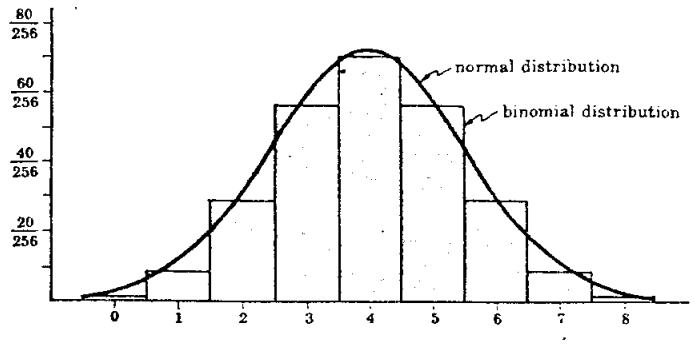
Here X^* is the standardized random variable corresponding to X, and hence X^* has the standard normal distribution N(0, 1).

NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION. CENTRAL LIMIT THEOREM

The binomial distribution P(k) = b(k; n, p) is closely approximated by the normal distribution providing n is large and neither p nor q is close to zero. This property is indicated in the following diagram where we have chosen the binomial distribution corresponding to n = 8 and $p = q = \frac{1}{2}$.

k	0	1	2	3	4	5	6	7	8
P(k)	$\frac{1}{256}$	$\frac{8}{256}$	$\frac{28}{256}$	$\frac{56}{256}$	$\frac{70}{256}$	$\frac{56}{256}$	$\frac{28}{256}$	$\frac{8}{256}$	$\frac{1}{256}$

Binomial distribution with n = 8 and $p = q = \frac{1}{2}$



Comparison of the binomial and normal distributions

The above property of the normal distribution is generalized in the Central Limit Theorem which follows. The proof of this theorem lies beyond the scope of this text.

Central Limit Theorem 6.4 : Let $X_1, X_2, \ldots$ be a sequence of independent random variables with the same distribution with mean μ and variance σ^2.

Let $$Z_n = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma}$$

Then for any interval $\{a \leqslant x \leqslant b\}$,

$$\lim_{n \to \infty} P(a \leqslant Z_n \leqslant b) = P(a \leqslant \phi \leqslant b)$$

where ϕ is the standard normal distribution.

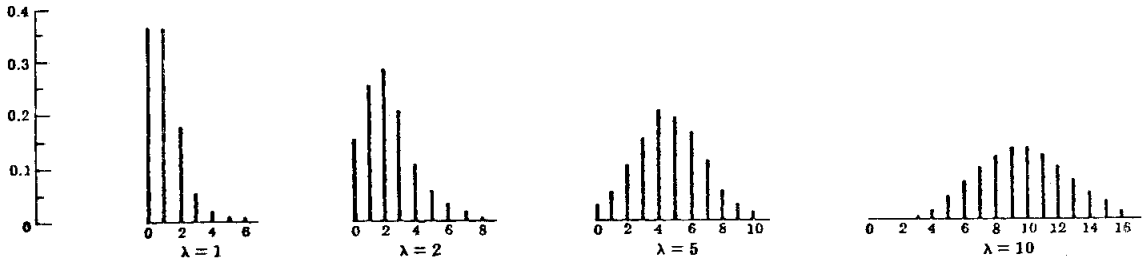
Recall that we called $\bar{S}_n = (X_1 + X_2 + \ldots + X_n)/n$ the sample mean of the random variables $X_1, \ldots, X_n$. Thus Z_n in the above theorem is the standardized sample mean. Roughly speaking, the central limit theorem says that in any sequence of repeated trials the standardized sample mean approaches the standard normal curve as the number of trials increase.

POISSON DISTRIBUTION

The Poisson distribution is defined as follows :

$$p(k ; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$ is some constant. This countably infinite distribution appears in many natural phenomena, such as the number of telephone calls per minute at some switchboard, the number of misprints per page in a large text, and the number of α particles emitted by a radioactive substance. Diagrams of the Poisson distribution for various values of λ follow.



Poisson distribution for selected values of λ

Properties of the Poisson distribution follow :

Theorem 6.5 :

Poisson distribution	
Mean	$\mu = \lambda$
Variance	$\sigma^2 = \lambda$
Standard deviation	$\sigma = \sqrt{\lambda}$

Although the Poisson distribution is of independent interest, it also provides us with a close approximation of the binomial distribution for small k provided that p is small and $\lambda = np$.

This is indicated in the following table.

k	0	1	2	3	4	5
Binomial	.366	.370	.185	.0610	.0149	.0029
Poisson	.368	.368	.184	.0613	.0153	.00307

Comparison of Binomial and Poisson distributions
with n = 100, p = 1/100 and $\lambda = np = 1$

MULTINOMIAL DISTRIBUTION

The binomial distribution is generalized as follows. Suppose the sample space of an experiment is partitioned into, say, s mutually exclusive events $A_1, A_2, \ldots, A_s$ with respective probabilities $p_1, p_2, \ldots, p_s$. (Hence $p_1 + p_2 + \ldots + p_s = 1$.) Then :

Theorem 6.6 : In n repeated trials, the probability that A_1 occurs k_1 times, A_2 occurs k_2 times, ..., and A_s occurs k_s times is equal to

$$\frac{n!}{k_1!\, k_2! \ldots k_s!}\, p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s}$$

where $k_1 + k_2 + \ldots + k_s = n$.

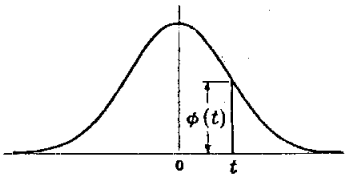
The above numbers form the so-called *multinomial distribution* since they are precisely the terms in the expansion of $(p_1 + p_2 + \ \ldots \ + p_s)^n$. Observe that if s = 2 then we obtain the binomial distribution, discussed at the beginning of the chapter.

Example 6.4 : A fair die is tossed 8 times. The probability of obtaining the faces 5 and 6 twice and each of the others once is

$$\frac{8!}{2!\,2!\,1!\,1!\,1!\,1!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{35}{5832} \approx .006$$

STANDARD NORMAL CURVE ORDINATES

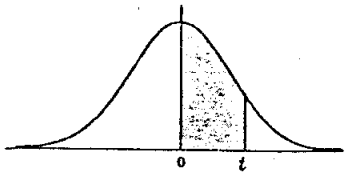
This table gives values $\phi(t)$ of the standard normal distribution ϕ at $t \geq 0$ in steps of 0.01.



t	0	1	2	3	4	5	6	7	8	9
0.0	.3989	.3989	.3989	.3988	.3986	.3984	.3982	.3980	.3977	.3973
0.1	.3970	.3965	.3961	.3956	.3951	.3945	.3939	.3932	.3925	.3918
0.2	.3910	.3902	.3894	.3885	.3876	.3867	.3857	.3847	.3836	.3825
0.3	.3814	.3802	.3790	.3778	.3765	.3752	.3739	.3725	.3712	.3697
0.4	.3683	.3668	.3653	.3637	.3621	.3605	.3589	.3572	.3555	.3538
0.5	.3521	.3503	.3485	.3467	.3448	.3429	.3410	.3391	.3372	.3352
0.6	.3332	.3312	.3292	.3271	.3251	.3230	.3209	.3187	.3166	.3144
0.7	.3123	.3101	.3079	.3056	.3034	.3011	.2989	.2966	.2943	.2920
0.8	.2897	.2874	.2850	.2827	.2803	.2780	.2756	.2732	.2709	.2685
0.9	.2661	.2637	.2613	.2589	.2565	.2541	.2516	.2492	.2468	.2444
1.0	.2420	.2396	.2371	.2347	.2323	.2299	.2275	.2251	.2227	.2203
1.1	.2179	.2155	.2131	.2107	.2083	.2059	.2036	.2012	.1989	.1965
1.2	.1942	.1919	.1895	.1872	.1849	.1826	.1804	.1781	.1758	.1736
1.3	.1714	.1691	.1669	.1647	.1626	.1604	.1582	.1561	.1539	.1518
1.4	.1497	.1476	.1456	.1435	.1415	.1394	.1374	.1354	.1334	.1315
1.5	.1295	.1276	.1257	.1238	.1219	.1200	.1182	.1163	.1145	.1127
1.6	.1109	.1092	.1074	.1057	.1040	.1023	.1006	.0989	.0973	.0957
1.7	.0940	.0925	.0909	.0893	.0878	.0863	.0848	.0833	.0818	.0804
1.8	.0790	.0775	.0761	.0748	.0734	.0721	.0707	.0694	.0681	.0669
1.9	.0656	.0644	.0632	.0620	.0608	.0596	.0584	.0573	.0562	.0551
2.0	.0540	.0529	.0519	.0508	.0498	.0488	.0478	.0468	.0459	.0449
2.1	.0440	.0431	.0422	.0413	.0404	.0396	.0387	.0379	.0371	.0363
2.2	.0355	.0347	.0339	.0332	.0325	.0317	.0310	.0303	.0297	.0290
2.3	.0283	.0277	.0270	.0264	.0258	.0252	.0246	.0241	.0235	.0229
2.4	.0224	.0219	.0213	.0208	.0203	.0198	.0194	.0189	.0184	.0180
2.5	.0175	.0171	.0167	.0163	.0158	.0154	.0151	.0147	.0143	.0139
2.6	.0136	.0132	.0129	.0126	.0122	.0119	.0116	.0113	.0110	.0107
2.7	.0104	.0101	.0099	.0096	.0093	.0091	.0088	.0086	.0084	.0081
2.8	.0079	.0077	.0075	.0073	.0071	.0069	.0067	.0065	.0063	.0061
2.9	.0060	.0058	.0056	.0055	.0053	.0051	.0050	.0048	.0047	.0046
3.0	.0044	.0043	.0042	.0040	.0039	.0038	.0037	.0036	.0035	.0034
3.1	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026	.0025	.0025
3.2	.0024	.0023	.0022	.0022	.0021	.0020	.0020	.0019	.0018	.0018
3.3	.0017	.0017	.0016	.0016	.0015	.0015	.0014	.0014	.0013	.0013
3.4	.0012	.0012	.0012	.0011	.0011	.0010	.0010	.0010	.0009	.0009
3.5	.0009	.0008	.0008	.0008	.0008	.0007	.0007	.0007	.0007	.0006
3.6	.0006	.0006	.0006	.0005	.0005	.0005	.0005	.0005	.0005	.0004
3.7	.0004	.0004	.0004	.0004	.0004	.0004	.0003	.0003	.0003	.0003
3.8	.0003	.0003	.0003	.0003	.0003	.0002	.0002	.0002	.0002	.0002
3.9	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0001	.0001

STANDARD NORMAL CURVE AREAS

This table gives areas under the standard normal distribution ϕ between 0 and $t \geq 0$ in steps of 0.01.



t	0	1	2	3	4	5	6	7	8	9
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0754
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2258	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2996	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

VALUES OF $e^{-\lambda}$										
λ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$e^{-\lambda}$	1.000	.905	.819	.741	.670	.607	.549	.497	.449	.407
λ	1	2	3	4	5	6	7	8	9	10
$e^{-\lambda}$	.368	.135	.0498	.0183	.00674	.00248	.000912	.000335	.000123	.000045

Table 6.3

ERRORS OF MEASUREMENT[2]

A measurement whose accuracy is completely unknown has no use whatever. It is therefore necessary to know how to estimate the reliability of experimental data and how to convey this information to others.

1. Classification of Errors

Errors of measurement are conveniently divided into several types. *Systematic* errors are the same for each observation with a given apparatus and method or are some definite function of the value of the quantity being observed. They may be due to the fact that the design of the apparatus was based on equations which are imperfect approximations to the actual situation. Thus Millikan had to correct his measurements of the electronic charge for deviations from Stokes' law of fall as applied to his tiny oil drops.

The apparatus may contain sources of systematic error. For example, scales may be imperfectly graduated or incorrectly aligned or positioned. The two arms of a chemical balance may differ, introducing a fixed percentage error unless suitable precautions are taken.

A result may involve the use of auxiliary quantities, which, if incorrect, will introduce error. Thus most methods of determining Planck's constant h require a knowledge of the electronic chagre e.

The most important of these categories is probably the first, the deviation of the real system from the laws assumed for it. An apparatus seldom measures the quantity it is supposed to; much more often it measures something which is assumed to be related to the quantity in question by a known law. Thus most of the methods of measuring the velocity of light in free space really measure the group velocity of light in air. There have been controversies about the relationship between these.

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[2]Wilson, Jr., E.B. *An Introduction to Scientific Research,* 1st ed., New York, McGraw-Hill, 1952, p. 232-246.

Personal Errors. In many types of measurement, such as reading scales and setting cross hairs, in which personal judgment enters, different individuals will often obtain systematically different results. One person may tend, for example, to be consistently later than another in actuating a stop watch for a given event. These *personal* errors may be detected by having several observers make the same measurement, but it is more difficult to decide which is nearest to the truth unless known standards are available. In the same general category are all the extremely prevalent number prejudices which almost everyone displays. These are so strong that they not only markedly influence the frequency with which the different digits occur in the estimation of tenths of a division on a scale but also may falsify results which simply need to be read directly and copied in a notebook. Each observer should take a long series of results obtained by himself on a slide rule, say, and examine the distribution of estimated digits. These may be tested by the χ^2 test (Sec. 8.10) if there is any doubt that selective bias has been at work.

Personal errors are most likely to creep in if the meaning of the readings can be directly interpreted by the observer in terms of a theory in which he has an interest. Persons of the highest probity will often unconsciously read a little high or a little low from this cause. More flagrant cases of subconscious bias usually concern nonnumerical observations. Eminent scientists have reported observations which a simple theory of the instruments used showed could not have been possible.

As has been previously emphasized (Sec. 4.6), it is the duty of every scientist to take precautions which will minimize this type of error. It is not sufficient to assert that they occur only with other people.

Mistakes. Such errors as those of arithmetic, misplacement of a decimal point, reversal of a sign, transposition of figures, recording of a wrong integer, the reading of a scale backward, or the use of the wrong scale are called mistakes. They can be minimized by obvious means. Improperly designed apparatus, with complicated types of scales, can be the origin of so many errors of this kind as to make it important to insist on simplicity at these points.

Assignable Causes. Every measurement is affected by a large number of variables. The experimenter aims to control, if possible, those which have important effects, leaving uncontrolled only the large number which individually influence the results very little. Usually this is a very difficult task, and often there remain a small number of variables which are not controlled and yet still have important effects. These have been called *assignable causes* which if not located and either eliminated or otherwise taken into account will interfere seriously with the accuracy and certainty of the results. For example, in Anderson's measurements of the velocity of light, he detected such a variation and finally located its cause. The two beams of light impinged on a photocell from slightly different directions so that the electrons ejected by the light took different paths and required slightly

different times to reach the collector. Anderson was unable to overcome this cause of variation, but his location of it led Bergstrand to a modified design which avoided this difficulty and gave much improved values. The location of errors of this type is considered in some detail in Secs. 9 and 10.

Random Errors. These are the variations due to the working of a number of uncontrolled variables, each of whose effects is individually small. As their name implies, successive observations of the same quantity should form a random sequence (see Secs. 9.9 and 10.2) if only random errors are acting. It is considered that there exists a hypothetical universe or population (Sec. 8.5), made up of the results which would be obtained if the given observation were repeated a very large number of times, under conditions such that assignable causes of error were excluded. Then any actual finite set of observations is regarded as a random sample from this population.

In practice this hypothesis of randomness can be accepted only when no more plausible explanation of the observed pattern of results is available. Very frequently, as discussed in Sec. 9.9, close study shows the existence of assignable causes which can be located and perhaps eliminated. However, when, and only when, randomness is the most reasonable characterization of the results, it becomes possible to apply statistical analysis to the data. These arguments lead to ways of reducing the effect of random errors and of estimating the uncertainty which they introduce. Ordinary statistical procedures are not applicable when the errors are not random. Modern statistical methods are based on very different fundamental views from those in vogue even a decade or two ago, and these affect the treatment of random errors.

2. The Normal Law of Error

It has been usual to assume that random errors are distributed according to the *normal law of error.* Justification for this assumption will be discussed later. This law states that the fraction dN/N of the population of observations whose values lie in the range x to x + dx is given by

$$\frac{dN}{N} = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx \qquad (1)$$

Here μ and σ are the two *population parameters* which specify the distribution. The quantity μ is the *mean* of the entire population (not just that of the finite sample observed), and σ, called the *standard deviation* of the population, is a measure of the spread of the population. By introducing the concept of *true errors,* defined by

$$e = x - \mu$$

i.e., the deviations from the population mean, the distribution becomes

$$\frac{dN}{N} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\epsilon^2/2\sigma^2} d\epsilon \tag{2}$$

Finally, if these errors are measured in units of the population standard deviation σ, that is, in terms of the variable

$$y = \frac{\epsilon}{\sigma} \tag{3}$$

then

$$\frac{dN}{N} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right) dy \tag{4}$$

a form which depends on the single variable y. Figure 9.1 shows a plot of the distribution. The meaning of this equation is that the fraction of the population with values of y between any specified limits is given by the area under the curve between those limits.

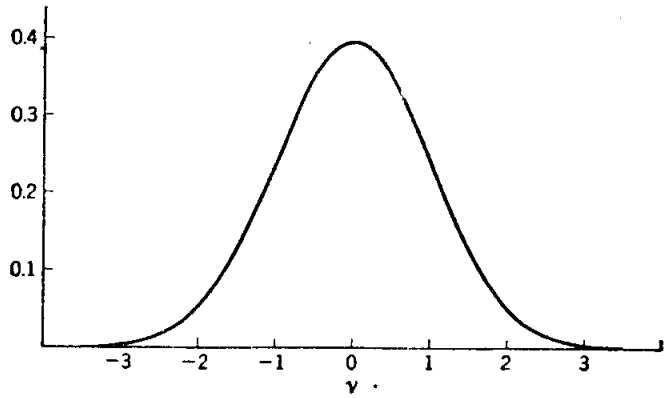

FIG. 9.1. Graph of the normal distribution law $(1/\sqrt{2\pi})\exp(-\frac{1}{2}y^2)$ as a function of $y = (x - \mu)/\sigma$.

It is very important to maintain a clear distinction between the parameters μ and σ, which characterize the infinite population of possible observations, and the properties of a particular finite *sample* from this population. For example, $\bar{x}$, the sample mean, and

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}} \tag{5}$$

the *sample standard deviation* for n observations, should not be confused with μ and σ. Greek letters are often used for population parameters to emphasize the difference. The choice of the divisor $n-1$ instead of n (which is often used) in the definition of s will be explained later.

Confidence Interval for μ When σ Is Known. In many experiments, at least a rough value is known for the population standard deviation σ (a measure of the "experimental error"), either by compounding estimated components of error (Sec. 9.11) or from previous experiments. Then a measurement is made, yielding a result x. It is realized that it is highly

unlikely that x will exactly equal μ (which is unknown), but it is also unlikely that it will differ from μ by many multiples of σ. What kind of statement can be made about the value of μ from this one observation x, together with three assumptions : that the measurement was a *random* selection from a *normal* population of known σ?

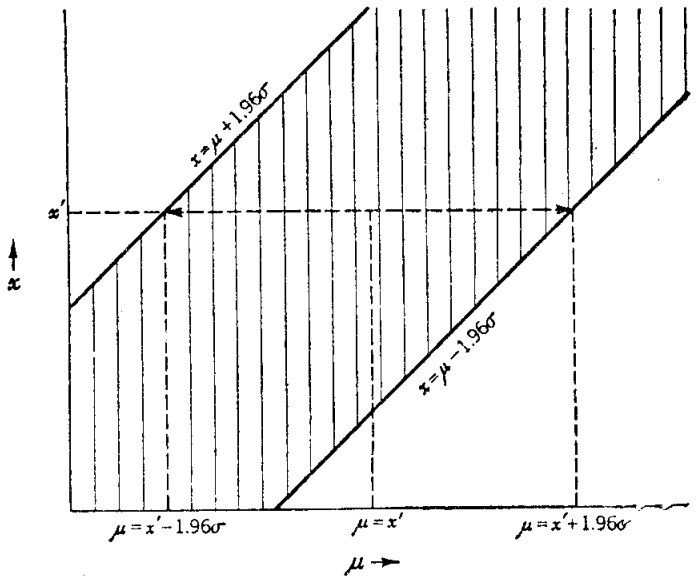

FIG. 9.2. Construction of 95 per cent confidence intervals for population mean μ when standard deviation σ is known. Normal population.

This is an application of the theory of confidence intervals discussed in Secs. 7.4 and 8.4. With the given σ and some assumed value of μ, plot the normal curve as a function of x. Now suppose that it is agreed to operate at a confidence level of 95 per cent. Then cut off 2.5 per cent of the area under the curve at each tail, leaving a range of x centered about μ. Calculation of the area shows that the range $\mu - 1.96\sigma < x < \mu + 1.96\sigma$ will contain 95 per cent of the observations. Make the same calculation for each possible value of μ. Construct a diagram with μ as ordinate and x as abscissa (Fig. 9.2). On this, plot the lines $x = \mu \pm 1.96\sigma$. These enclose a (shaded) band which actually extends an infinite distance in each direction. By the way in which this band has been constructed, it will enclose 95 per cent of all possible observations in all possible normal populations with the given value of σ, regardless of the value of μ. Now consider the situation in which a particular value of x has been observed, σ being known, but μ not known. Draw a horizontal dashed line on Fig. 9.2 at the level of the observed value of x. It then is reasonable to assert that μ lies somewhere in the range $x - 1.96\sigma$ to $x + 1.96\sigma$. This is not an expression of certainty but one made at a 95 per cent level of confidence. This means that, in the long run, if similar statements were made about similar observations, only 5 per cent of them would be incorrect. That this is true follows from the fact that the shaded region contains 95 per cent of all possible members of the normal populations with all values of μ but given σ. If the assertion is made that a

particular observation must have come from the shaded area, it will be right in the long run 95 per cent of the time.

Naturally other values of the level of confidence can be chosen, according to the importance of the risk of not covering the true value of μ (here 5 per cent) as compared with the desirability of making a more definite statement regarding the value of μ. Table 9. 1 gives the multiples of σ appropriate for other levels of confidence.

TABLE 9.1 CONFIDENCE INTERVALS FOR VARIOUS LEVELS OF CONFIDENCE. NORMAL POPULATION WITH KNOWN σ

Confidence Level	Confidence Interval
.50	$x - 0.674\sigma < \mu < x + 0.674\sigma$
.80	$x - 1.282\sigma < \mu < x + 1.282\sigma$
.90	$x - 1.645\sigma < \mu < x + 1.645\sigma$
.95	$x - 1.9600 < \mu < x + 1.960\sigma$
.99	$x - 2.5760 < \mu < x + 2.5160$
.999	$x - 3.291\sigma < \mu < x + 3.291\sigma$

Unsymmetrical Confidence Intervals. The critical reader will ask why the particular 95 per cent area used in Fig. 9.2 was chosen. Why not, for example, exclude 5 per cent of the samples in just one tail of the curve or divide off the $x\mu$ plane in any of an infinite number of ways which would shade a region containing 95 per cent of all possible members? One reason for the particular choice used is that it gives the shortest range, or *confidence interval,* for μ. However, situations could well arise in which it would be sensible to make a different choice. Suppose that the cost of overestimating μ were four times greater than the cost of underestimating it. Then it would be appropriate to use the unsymmetrical confidence interval $x - 2.3260$ to $x + 1.751\sigma$, which still includes 95 per cent of all the populations but excludes 4 per cent of the tail on the low side and only 1 per cent on the high side. This is clearly slightly longer than the symmetrical choice.

Case of Several Observations. Usually a measurement is repeated several times and the mean x calculated. Under the same assumptions as before, it is easy to calculate confidence intervals for the population mean μ when what is given is the mean $\bar{x}$ for a random sample of n from a normal population of known σ. For 95 per cent confidence, one obtains, as will be shown,

$$\frac{\bar{x} \quad 1.960}{\sqrt{n}} < \mu < \frac{\bar{x} + 1.96\sigma}{\sqrt{n}} \tag{6}$$

The other levels of confidence can be obtained similarly by substituting the numerical coefficients given in Table 9.1. In other words, the use of the mean of a sample of n instead

of just one observation closes down the confidence interval by a factor $\sqrt{n}$ so that, on this basis, the mean of 100 observations should be ten times as precise as a single observation. This principle is very important, but it has severe practical limitations, which will be discussed later, in Sec. 9.6.

To derive Eq. (6) above, it is necessary to consider the joint distribution of n observations. When the population is normal, the probability dP that the first observation is in dx_1 at x_1, the second in dx_2 at x_2, etc., under the assumption that the separate observations are quite independent of one another, is the product of the separate probabilities, or

$$dP = (2\pi)^{-n/2}\sigma^{-n}\exp\left[\frac{-\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right]dx_1\,dx_2\ldots dx_n \tag{7}$$

By a simple change of variables, followed by integration over all the new variables except $\bar{x}$ and s, defined by

$$\bar{x} = \frac{\Sigma x_i}{n} \tag{8}$$

and

$$s^2 = \frac{\Sigma(x_i-\bar{x})^2}{n-1} \tag{9}$$

there is obtained

$$dP = C\left|e^{-n(\bar{x}-\mu)^2/2\sigma^2}d\bar{x}\right|\left|s^{n-2}e^{-(n-1)s^2/2\sigma^2}\,ds\right| \tag{10}$$

C is a normalizing constant independent of $\bar{x}$ and s. This equation shows how the sample mean x and sample standard deviation s are distributed. It is seen that they are entirely independent of one another, so that a given sample of n observations may scatter very widely (large s) and yet have its mean $\bar{x}$ close to μ. Furthermore, integration over all values of s (0 to ∞) yields

$$dP = (2\pi)^{-\frac{1}{2}}\left(\frac{n}{\sigma^2}\right)^{\frac{1}{2}}e^{-n(\bar{x}-\mu)^2/2\sigma^2}\,d\bar{x} \tag{11}$$

for the distribution of the mean $\bar{x}$ alone. This shows that $\bar{x}$ is normally distributed with standard deviation $\sigma/\sqrt{n}$, which is the result required to give Eq. (6) for the confidence interval.

If it is certain that the data can be considered as a random sample from a normal population of known standard deviation σ, then the only feature of the data of any importance is its mean $\bar{x}$. However unrepresentative the sample standard deviation s may be, or the order in which the values were obtained, these can only be viewed as perhaps surprising but not as the basis for any action.

In practice, no such certainty regarding the randomness, the normality of the population, or the standard deviation σ ever exists. At most the observer may have a high degree of confidence in the approximate conformity of his data to these conditions. This confidence is not ironclad and might be shaken if the standard deviation of the sample were enormously greater than σ, for example. This intermediate situation will be treated later,

but first a discussion will be given of the other extreme case, the one in which no prior knowledge whatever of σ is available.

Confidence Interval When σ Is Unknown. In some situations no prior knowledge of the population standard deviation is available. Under these conditions (which very rarely occur in this extreme form) a single observation x gives essentially no information about the population mean μ at all. However, a sample of $n > 1$ observations does provide information since the spread, as measured by s, is an indication of the value of σ.

The joint distribution of $\bar{x}$ and s was given in Eq. (10). Transformation from these variables to the new set

$$t = \frac{(\bar{x})-\mu)\sqrt{n}}{s} \quad \text{and} \quad u = \frac{s^2}{\sigma^2} \tag{12}$$

followed by integration over all values of u, leads (see the Notes) to the so-called Student's t distribution,

$$dP = C'\left(1+\frac{t^2}{n-1}\right)^{-n/2}dt \tag{13}$$

C' being a normalizing constant dependent on n. This function has been tabulated for small values of n. For large values it rapidly approaches a normal distribution. This distribution involves only the sample properties $\bar{x}$ and s and the population mean μ. Since σ has been eliminated, it is possible to use the t distribution to obtain confidence intervals of the case in which σ is unknown. These are given by

$$\bar{x}-\frac{k}{\sqrt{n}}s < \mu < \bar{x}+\frac{k}{\sqrt{n}}s \tag{14}$$

where $\bar{x}$ is the sample mean, s is the sample standard deviation,

$$s = \left[\frac{\sum_i (x_i-\bar{x})^2}{n-1}\right]^{\frac{1}{2}} \tag{15}$$

and k is obtained from an integration of the t distribution. Some values are given in Table 9.2.

Comparison of Two Types of Intervals. It is very important that the true meaning of these confidence intervals be thoroughly understood. To illustrate this point, an idealized experiment was performed. Thirty-two pairs of numbers were drawn randomly from a normal distribution with $\mu = 0$ and $\sigma = 1$. For each pair the sample mean x was calculated and plotted as a short horizontal bar in Fig. 9.3. With the known σ of unity, 50 per cent confidence limits were computed, giving intervals of fixed length $0.95\sigma = 0.95$. These are shown as solid vertical lines centered at the means $\bar{x}$ With the use of Eq. (14), confidence limits were also calculated for the case in which the value of σ is not known but in which the sample standard deviation s must be employed. Since the value of s is subject to rather violent fluctuations for samples of only two observations (or even for considerably larger values of n), the lengths of these intervals vary greatly. They are shown as dashed

Table 9.2. Coefficients k for Use in Calculating Confidence Limits with Eq. (14). Entries Are Also Critical Values t, for Student t Distribution*

[See Eq. (4), Sec. 9.10]

Confidence level

Degrees of freedom	No. observations	.50	.80	.90	.95	.99
1	2	1.00	3.08	6.31	12.71	63.66
2	3	.82	1.89	2.92	4.30	9.93
3	4	.77	1.64	2.35	3.18	5.84
4	5	.74	1.53	2.13	2.78	4.60
5	6	.73	1.48	2.02	2.57	4.03*
6	7	.72	1.44	1.94	2.45	3.71
7	8	.71	1.42	1.90	2.37	3.50
8	9	.71	1.40	1.86	2.31	3.36
9	10	.70	1.38	1.83	2.26	3.25
10	11	.70	1.37	1.81	2.23	3.17
11	12	.70	1.36	1.80	2.20	3.11
12	13	.70	1.36	1.78	2.18	3.06
13	14	.69	1.35	1.77	2.16	3.01
14	15	.69	1.35	1.76	2.15	2.98
15	16	.69	1.34	1.75	2.13	2.95
∞	∞	.67	1.28	1.65	1.96	2.58

* Abridged from Table IV of Fisher and Yates, *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver & Boyd, Ltd., Edinburgh and London, by permission of the authors and publishers.

vertical lines, also centered on $\bar{x}$. It will be seen that very nearly half (16/32 and 17/32) of each kind of confidence intervals do overlap the true population mean $\mu = 0$. The lengths were calculated on a theory designed to make this result true in the long run. Note that the two types of intervals achieve this average result in two different ways. When σ is known, the interval is of fixed length but its center is subject to the inevitable sampling fluctuations of x. When σ is unknown, the length of the interval also fluctuates because s does, but the lengths are such that on the average half of them still cover the true value, though it is not always the same ones which are successful. Since one method uses more information than the other, it is to be expected that the *average* length of the Student-type intervals would be longer than the other type and this is in fact the case. Therefore if σ is definitely known, it should be used and s ignored, because, on the average, more definite statements can be made at the same level of confidence.

The figure also illustrates two points on which the classical treatments of the theory of errors were often incorrect. In the first place, the usual practice was to estimate σ by the use of s and then to employ the fixed range based on this estimate of σ. For small samples this gives ranges which are considerably too short on the average to cover

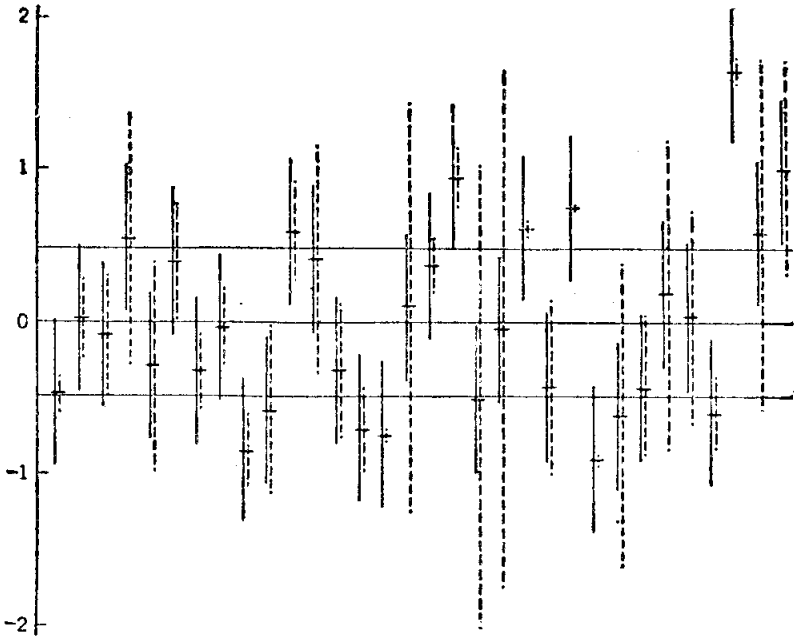

FIG. 9.3. A set of 32 samples of two, drawn at random from a normal distribution of mean zero and $\sigma = 1$. Horizontal bars are means, vertical solid lines are 50 per cent confidence intervals calculated with known σ, vertical dashed lines are 50 per cent confidence intervals calculated with the Student method which does not use a knowledge of σ. About half of each type overlap the true value (0). The pair of horizontal lines are calculated from the known μ and σ so as to contain half the means in the long run (actually include 16 out of 32).

the true value the desired fraction of the time. Student's work corrects this error, which is not serious for large n, say $n > 15$.

The second error is one of basic viewpoint. The reader was often led to the idea that he could use the sample s to determine limits within which half of all future means would lie. The fluctuations of s from sample to sample, as well as those of x, make this a very hazardous enterprise, unless n is quite large. To take an unfavorable case, suppose such an inference had been made from the data of the first pair of Fig. 9.3. Instead of half of the other thirty-one means lying within the dotted range calculated from this pair, only four are included.

The correct viewpoint, as has been stated, is that if confidence intervals are repeatedly calculated and each time the interval is asserted to overlap μ, these assertions will be right in the long run a certain fraction of the time—the confidence level adopted—provided the other assumptions required are satisfied. A high level of confidence goes with a long interval, a low level with a shorter interval.

Situation When σ Is Known Roughly. In practice it usually occurs that a rough a priori knowledge of σ is available, but not an accurate value of great certainty. It is then possible to combine the prior knowledge of σ with the observed s of the sample to

calculate a confidence interval which makes use of all the available information. The basis for this calculation is Bayes' theorem of Sec. 10.1.

This procedure is not ordinarily used. Suppose that s comes out much higher than the sampling distribution of s, in terms of the estimated σ, would make probable. [The second factor of Eq. (10) governs this distribution.] Then it is rather likely that the estimate of σ is wrong, or that the apparatus is not in control, or that a mistake has been made.

However, it would be reasonable to make a rough, intuitive application of Bayes' theorem in those cases in which σ was only roughly known, by adopting a confidence interval somewhere in between that based on σ and that based on s, according to some rule decided upon in advance of the experiment.

Other Estimates of Precision. The confidence-interval approach is probably the most satisfactory way at present available of indicating the estimated reliability of an average, in so far as random errors are concerned. It is often desired, however, to make an estimate of the spread of the parent population. The best measure of this spread is σ^2, the square of the population standard deviation, and this quantity is called the population *variance*.

If the only information available is that contained in a sample of n observations, the best estimate of σ^2 is

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 \tag{16}$$

The divisor $n-1$ is used in order that the estimate will come out right in the long run when averaged over all samples of size n. An estimate which is correct on the average in this way is called an *unbiased* estimate. One advantage of Eq. (16) above is that it provides an estimate of σ^2 even when the parent population is not normal. As has been emphasized, a small sample cannot provide a very reliable estimate of σ^2 because of sampling fluctuations. For normal populations, however, s^2 provides the most *efficient* estimate of σ^2, that is, the one with the least sampling variation (in a certain sense). Whenever it is desired to get the most out of a sample, s^2 should therefore be the estimate used.

A short-cut method of calculating s^2 is very useful. Let a be any constant whatever. Then

$$\begin{aligned} s^2 &= \frac{1}{n-1} \sum [(x_i - a) - (\bar{x} - a)]^2 \\ &= \frac{1}{(n-1)} [\Sigma (x_i - a)^2 - 2(\bar{x} - a) \Sigma (x_i - a) + n(\bar{x} - a)^2] \end{aligned} \tag{17}$$

Because of the definition of $\bar{x}$, the last two terms combine, so that

$$s^2 = \frac{1}{n-1}\Sigma(x_i-a)^2 - \frac{n}{n-1}\overline{(x-a)}^2$$

$$= \frac{1}{n-1}\left\{\Sigma(x_i-a)^2 - \frac{1}{n}\left[\Sigma(x_i-a)\right]^2\right\} \tag{18}$$

Consequently, all observations x_i should first be reduced in size by subtracting an appropriately chosen constant a from each. Since variations in experimental measurements are often only a few times the least count, the reduced values will seldom exceed a hundred units. Their squares are then easily obtained from a small table. With this procedure, s^2 is not at all formidable to calculate. Table 9.3 gives a worked-out example of the calculation of s^2, for a set of values drawn at random from a normal population with $\mu = 10$ and $\sigma = 0.50$.

There are other ways of estimating σ (or σ^2). None are as effcient as s, but some are easier to calculate. For small samples (say $n < 11$) the *range* w is useful. This is the difference between the greatest and least values in the sample. It becomes a less and less efficient estimator as the sample size increases and is ordinarily not used for large samples, except for quick estimates and checking. In order to obtain an unbiased estimate of σ, it is necessary to multiply the range by a numerical factor, given in Table 9.4. These factors can be calculated by first obtaining a distribution function for the range and then averaging.

The range for the data in Table 9.3 is 1.56, which gives .507 as an estimate of σ, which is better than the value .511 obtained from s, but on the average s will give the more accurate estimate.

TABLE 9.3. AN EXAMPLE OF THE CALCULATION OF THE SAMPLE VARIANCE s^2 BY SHORT-CUT METHOD

Observations	(Observations − 10.00)	Squares	
10.68	.68	.4624	$\Sigma(x_i - a)^2 = 2.3560$
9.24	−.76	.5776	$(1/n)[\Sigma(x_i - a)]^2 = .0078$
10.42	.42	.1764	$s^2 = 9/2.3482$
10.80	.80	.6400	$\quad .2609$
9.83	−.17	.0289	$s = .511$
10.04	.04	.0016	(Cf. $\sigma = .500$)
9.49	−.51	.2601	$\bar{x} = 10.00 - .028$
9.75	−.25	.0625	$= .972$
9.79	−.21	.0441	(*Cf.* $\mu = 10.00$)
9.68	−.32	.1024	
	−.28	2.3560	

For small samples the mean deviation from the mean (average without regard to sign) is less efficient than the range and certainly harder to calculate. For large samples the mean deviation, multiplied by 1.253, gives an estimate of σ which is better than that provided by the range. However, by breaking the sample randomly into a number of equal parts and averaging the ranges of the parts, an easier estimate of σ can be made which is practically as efficient. Other methods are also available which are suitable for punched-card machines.

TABLE 9.4. MULTIPLIERS FOR CONVERTING RANGE w FOR A SAMPLE OF n INTO AN UNBIASED ESTIMATE OF THE POPULATION STANDARD DEVIATION σ. FOR NORMAL POPULATIONS ONLY

[From O.L. Davies and E.S. Pearson, *Suppl. J. Roy. Statistical Soc.*, 1, 76 (1934)]

n	2	3	4	5	6	7	8	9	10
Multiplier for range	0.886	.591	.486	.430	.395	.370	.351	.337	.325

Often a number of small samples are available which can be assumed to come from normal populations of the same σ but different means μ. Then it is possible to average the estimates of σ provided by each sample, in order to obtain a more reliable estimate from the whole set of data. Thus if all measurements with a given piece of apparatus (such as a measuring engine) are made in duplicate, the whole set of pairs can provide a quite accurate estimate of the error σ even though the estimates from the individual pairs fluctuate widely.

The most efficient way of estimating the variance σ^2 in such a case is with the formula

$$\frac{1}{N-k} \sum_{j=1}^{k} \sum_{i=1}^{n_j} (x_{ij} - \bar{x}_j)^2 \tag{19}$$

in which x_{ij} is the ith observation in the jth sample, $\bar{x}_j$ is the mean of the jth sample, with n_j members, and N is the total number of observations. There are k samples.

A Simple Substitute for t. The range w can be used instead of s in forming confidence intervals when the true variance σ^2 is unknown. This is much easier to calculate and turns out to be very nearly as efficient, even for large samples, as the use of t. However, all uses of the range are very susceptible to "wild" values.

For a given level of confidence, the confidence limits based on the range are given approximately by

$$\bar{x} - c_n w < \mu < \bar{x} + c_n w \tag{20}$$

in which $\bar{x}$ is the mean of a random sample of n values from a normal distribution, w is the difference between the greatest and least values in the sample, and c_n is a constant given in Table 9.5.

TABLE 9.5. VALUES OF c_n FOR CONFIDENCE INTERVALS BASED ON RANGE

[From E. Lord, *Biometrika*, 34, 41 (1947)]

Confidence level	90%	95%	98%	99%
2	3.157	6.353	15.910	31.828
3	0.885	1.304	2.111	3.008
4	0.529	0.717	1.023	1.316
5	0.388	0.507	0.685	0.843
6	0.312	0.399	0.523	0.628
7	0.263	0.333	0.429	0.507
8	0.230	0.288	0.366	0.429
9	0.205	0.255	0.322	0.374
10	0.186	0.230	0.288	0.333

3. Applicability of the Normal Law

Much controversy has raged over the general applicability of the normal law to errors of observation, and it seems certain that there is still a good deal left to be said. Many writers simply assume that it can always be used, others put forward theoretical arguments in its favor, while still others have presented sets of data which either did or did not support the law.

There can be little doubt regarding the qualitative correctness of the law for most cases of measurement error in which the scale divisions or class intervals are sufficiently fine. One expects and finds that the data support the idea of a continuous distribution with a single maximum and a monotonic falling off toward zero on either side. The curve generally appears to be approximately symmetrical, unless limited too closely on one side by zero. In practice, actual data seldom provide very firm information about the frequency of large errors because their occurrence is so rare. Usually the situation is such that extremely large errors are either impossible because of limitations in the length of scales, etc., or are rejected by the observer as being due to mistakes. The normal law, on the other hand, permits errors of any magnitude, but the larger ones would occur so infrequently that this is largely an academic question.

The theoretical argument for the normal law is based on the rapid approach to normality which can be demonstrated mathematically to occur when the error is due to the sum of a number of independent causes of about equal magnitude, each cause being distributed in any arbitrary manner provided only that each possesses a finite standard

deviation. Even if there are only four causes, each distributed binomially (Sec. 8.6), the result is hard to distinguish from a grouped normal distribution. Therefore, it does appear safe to use the normal law for observations in which is clear that four or five or more sources of error enter with about equal weight. This is true of many measurements which are systematically repeated several times with various interchanges and only the average used. It is also true of many measurements which involve making several independent adjustments for each observation, provided that these adjustments are all repeated each time and have roughly equal importance.

The reason why this theoretical justification is not applicable to all measurements is that there are many situations in which it is not obvious that there are a number of independent sources of error or, if so, that they are of roughly equal importance. If there is only one major source of error, it may be quite nonnormal. An example is the case of linear measurement with a scale, the last decimal being estimated by eye. With many observers, highly nonnormal distributions will be obtained because of strong number prejudices.

Experimental attempts to verify the normal law are not easy because of the very large number of observations required, which should be under comparable conditions. In order to obtain very much information about the distribution in the tails, 500 or more readings are needed. Few people have had the patience or time to acquire such data under constant conditions. Furthermore, it seems not to have been appreciated until recently that evidence is required that the observations are free from assignable causes, *i.e.*, are in statistical control (Sec. 9.9). Usually the order in which the observations were made is not known, so that tests for randomness cannot be applied. In a few cases where this is possible it is found that there were assignable causes, and therefore these data provide no test of the distribution law.

In a situation where a given measurement process has been established as a routine, so that hundreds or thousands of observations are made, it should be of value to test for normality. It is frequently customary in such measurements to require duplicate observations. Even though the specimens being measured differ in magnitude from one another, the pairs can be used to test the assumption that they came from normal distributions with the same standard deviation σ but different means μ. This analysis may be well worth while financially because a sure knowledge that the normal law applies would permit more certain inference to be drawn from the measurements. Otherwise a good deal of information contained in the data, and acquired at considerable money cost, is being thrown away.

Further, the differences between members of such pairs (or perhaps the per cent differences) can be tested for statistical control. Investigations of this kind can turn up all sorts of unsuspected defects in the procedure or equipment. It is important, however,

that two conditions be satisfied. The first is that the members of the pairs to truly independent. They will not be if the operator can have any memory or knowledge of the first reading when making the second. The record sheet or notebook should be provided with a half page designed to cover the first set, and the routine should be arranged so as to make it difficult for the operator to remember the first result. Also, if repeats are made in case of disagreements, the original results should not be erased. A second requirement is that the least count or smallest class interval used should be small compared with the spread of duplicate measurements. This is discussed in Sec. 9.5.

ชื่อ นามสกุล รหัส

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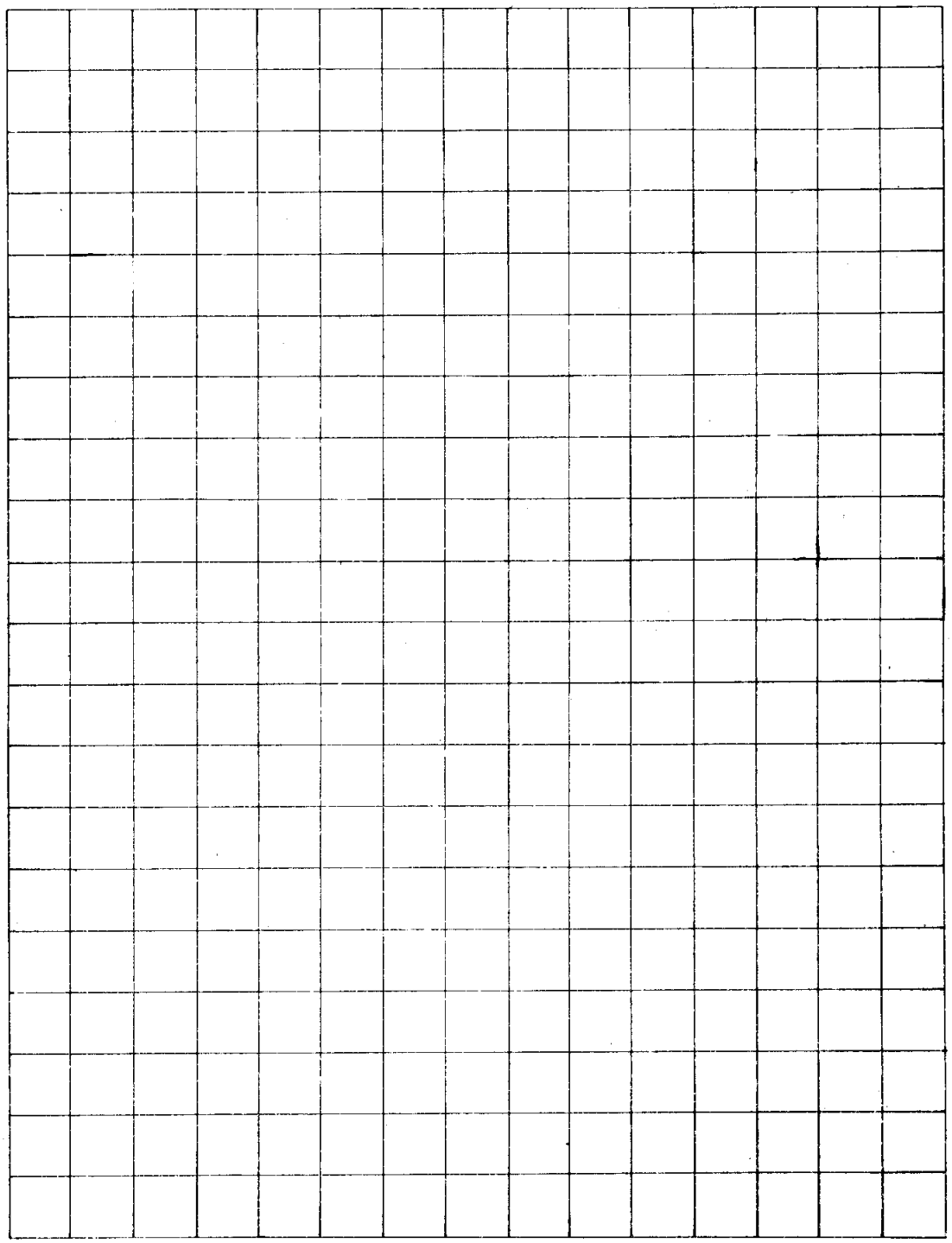

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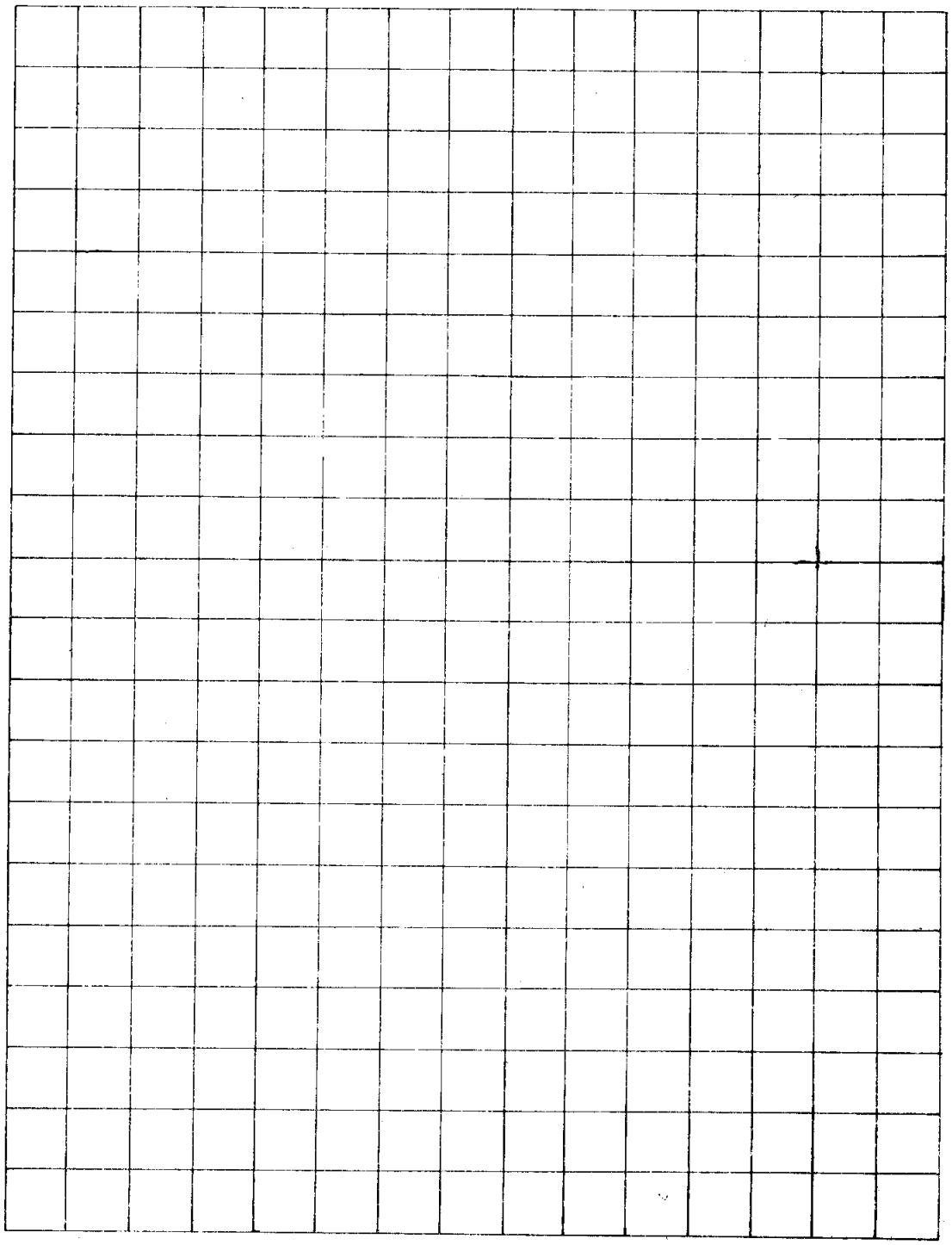

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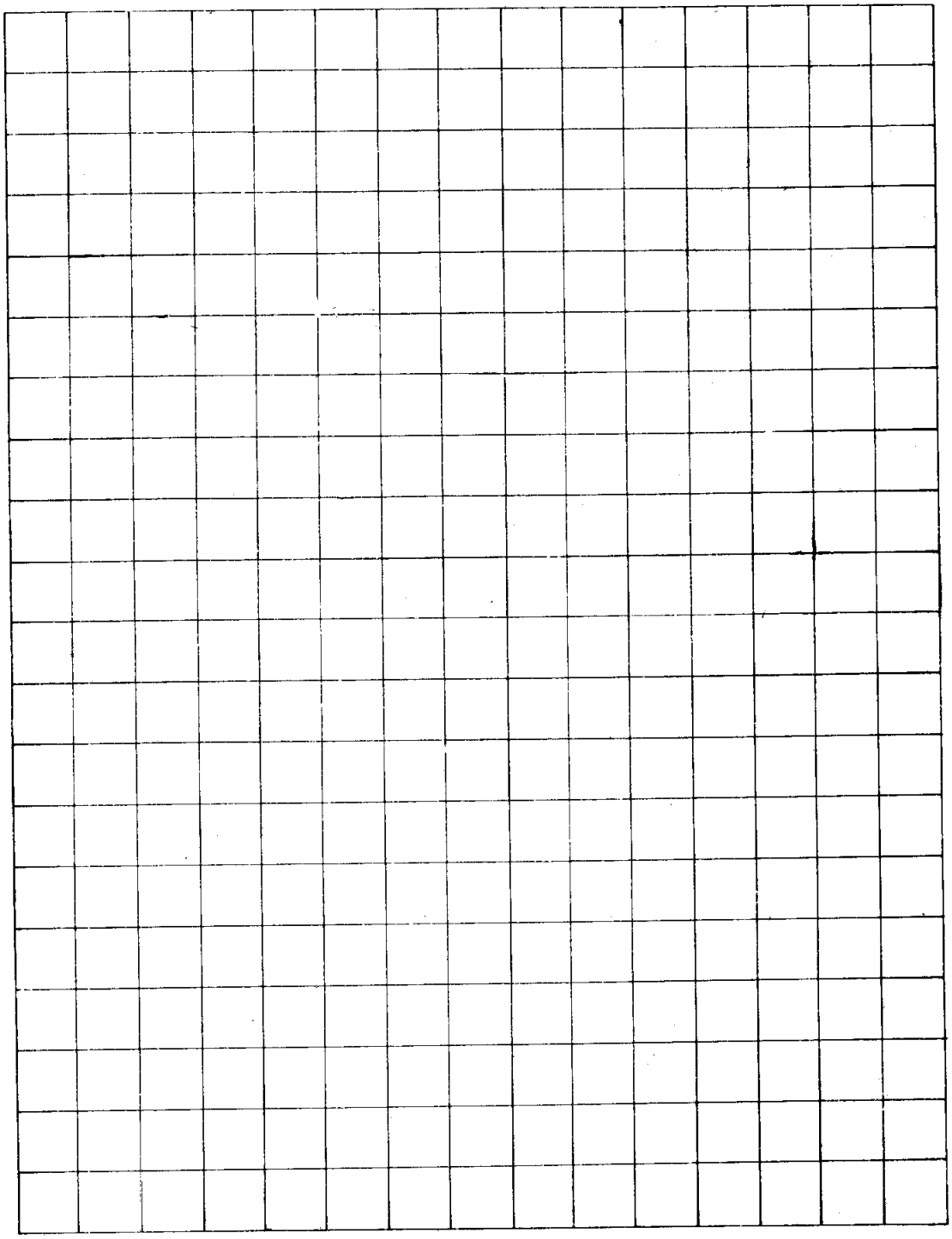

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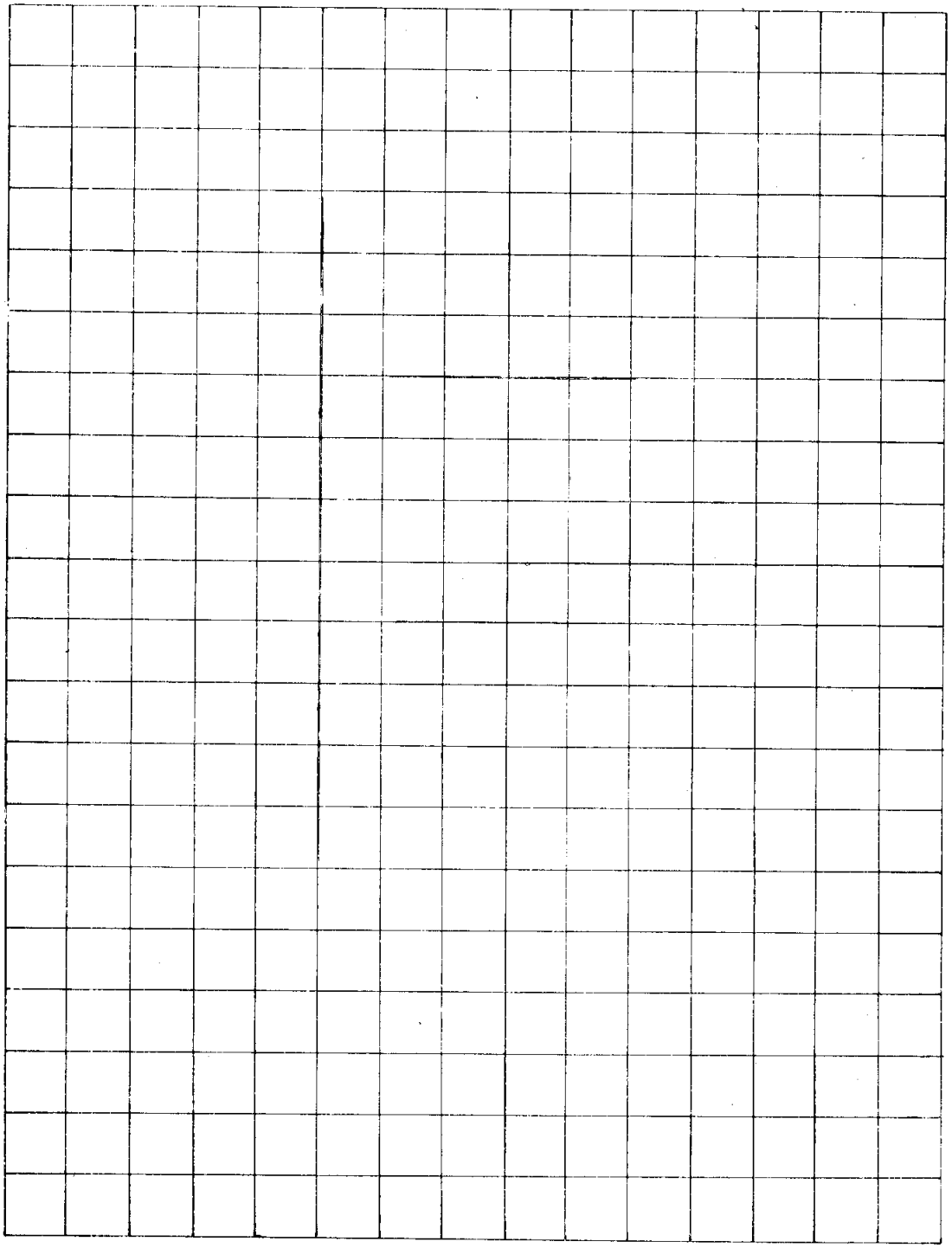

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