

ภาคผนวก 2

สูตรอินทิกรัล

Some Elementary Forms

1. $\int df(x) = \int f'(x) dx = f(x) + C.$
2. $\int a du = a \int du.$
3. $\int (du \pm dv \pm dw \pm \dots) = \int du + \int dv \pm \int dw \pm \dots$
4. $\int u^n du = \frac{u^{n+1}}{n+1} + C. \quad (n \neq -1)$
5. $\int \frac{du}{u} = \ln u + C.$

Rational Forms containing $a + bu$

See also the Binomial Reduction Formulas 96-104.

6. $\int (a + bu)^n du = \frac{(a + bu)^{n+1}}{b(n+1)} + C. \quad (n \neq -1)$
7. $\int \frac{du}{a + bu} = \frac{1}{b} \ln(a + bu) + C.$
8. $\int \frac{u du}{a + bu} = \frac{1}{b^2} [a + bu - a \ln(a + bu)] + C.$
9. $\int \frac{u^2 du}{a + bu} = \frac{1}{b^3} \left[\frac{1}{2} (a + bu)^2 - 2a(a + bu) + a^2 \ln(a + bu) \right] + C.$
10. $\int \frac{u du}{(a + bu)^2} = \frac{1}{b^2} \left[\frac{a}{a + bu} + \ln(a + bu) \right] + C.$
11. $\int \frac{u^2 du}{(a + bu)^2} = \frac{1}{b^3} \left[a + bu - \frac{a^2}{a + bu} - 2a \ln(a + bu) \right] + C.$
12. $\int \frac{u du}{(a + bu)^3} = \frac{1}{b^2} \left[-\frac{1}{a + bu} + \frac{a}{2(a + bu)^2} \right] + C.$

$$13. \int \frac{du}{u(a+bu)} = -\frac{1}{a} \ln \left(\frac{a+bu}{u} \right) + C.$$

$$14. \int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2} \ln \left(\frac{a+bu}{u} \right) + C.$$

$$15. \int \frac{du}{u(a+bu)^2} = \frac{1}{a(a+bu)} - \frac{1}{a^2} \ln \left(\frac{a+bu}{u} \right) + C.$$

Rational Forms containing $a^2 \pm b^2u^2$

$$16. \int \frac{du}{a^2 + b^2u^2} = \frac{1}{ab} \arctan \frac{bu}{a} + C.$$

$$17. \int \frac{du}{a^2 - b^2u^2} = \frac{1}{2ab} \ln \left(\frac{a+bu}{a-bu} \right) + C. \quad (a^2 > b^2u^2)$$

$$\int \frac{du}{b^2u^2 - a^2} = \frac{1}{2ab} \ln \left(\frac{bu-a}{(bu+a)} \right) + C. \quad (a^2 < b^2u^2)$$

$$18. \int u(a^2 \pm b^2u^2)^n du = \frac{(a^2 \pm b^2u^2)^{n+1}}{\pm 2b^2(n+1)} + C. \quad (n \neq -1)$$

$$19. \int \frac{u du}{a^2 \pm b^2u^2} = \frac{1}{\pm 2b^2} \ln(a^2 \pm b^2u^2) + C.$$

$$20. \int \frac{u^m du}{(a^2 \pm b^2u^2)^p} = \frac{u^{m-1}}{\pm b^2(m-2p+1)(a^2 \pm b^2u^2)^{p-1}} \\ - \frac{a^2(m-1)}{\pm b^2(m-2p+1)} \int \frac{u^{m-2} du}{(a^2 \pm b^2u^2)^p}$$

$$21. \int \frac{u^m du}{(a^2 \pm b^2u^2)^p} = \frac{u^{m+1}}{2a^2(p-1)(a^2 \pm b^2u^2)^{p-1}} \\ - \frac{m-2p+3}{2a^2(p-1)} \int \frac{u^m du}{(a^2 \pm b^2u^2)^{p-1}}$$

$$22. \int \frac{du}{u(a^2 \pm b^2u^2)} = \frac{1}{2a^2} \ln \left(\frac{u^2}{a^2 \pm b^2u^2} \right) + C.$$

$$23. \int \frac{du}{u^m(a^2 \pm b^2u^2)^p} = -\frac{1}{a^2(m-1)u^{m-1}(a^2 \pm b^2u^2)^{p-1}} \\ - \frac{\pm b^2(m+2p-3)}{a^2(m-1)} \int \frac{du}{u^{m-2}(a^2 \pm b^2u^2)^p}$$

$$24. \int \frac{du}{u^m(a^2 \pm b^2u^2)^p} = \frac{1}{2a^2(p-1)u^{m-1}(a^2 \pm b^2u^2)^{p-1}} + \frac{m+2p-3}{2a^2(p-1)} \int \frac{du}{u^m(a^2 \pm b^2u^2)^{p-1}}$$

Forms containing $\sqrt{a+bu}$

The integrand may be rationalized by setting $a+bu = v^2$. See also the Binomial Reduction Formulas 96-104.

$$25. \int u\sqrt{a+bu} du = -\frac{2(2a-3bu)(a+bu)^{3/2}}{15b^2} + C.$$

$$26. \int u^2\sqrt{a+bu} du = \frac{2(8a^2-12abu+15b^2u^2)(a+bu)^{3/2}}{105b^3} + C.$$

$$27. \int u^m\sqrt{a+bu} du = \frac{2u^m(a+bu)^{3/2}}{b(2m+3)} - \frac{2am}{b(2m+3)} \int u^{m-1}\sqrt{a+bu} du.$$

$$28. \int \frac{u du}{\sqrt{a+bu}} = -\frac{2(2a-bu)\sqrt{a+bu}}{3b^2} + C.$$

$$29. \int \frac{u^2 du}{\sqrt{a+bu}} = \frac{2(8a^2-4abu+3b^2u^2)\sqrt{a+bu}}{15b^3} + C.$$

$$30. \int \frac{u^m du}{\sqrt{a+bu}} = 2\frac{u^m\sqrt{a+bu}}{b(2m+1)} - \frac{2am}{b(2m+1)} \int \frac{u^{m-1} du}{\sqrt{a+bu}}$$

$$31. \int \frac{du}{u\sqrt{a+bu}} = \frac{1}{\sqrt{a}} \ln \left(\frac{\sqrt{a+bu} - \sqrt{a}}{\sqrt{a+bu} + \sqrt{a}} \right) + C, \text{ for } a > 0.$$

$$32. \int \frac{du}{u\sqrt{a+bu}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bu}{-a}} + C, \text{ for } a < 0.$$

$$33. \int \frac{du}{u^m\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(m-1)u^{m-1}} - \frac{b(2m-3)}{2a(m-1)} \int \frac{du}{u^{m-1}\sqrt{a+bu}}$$

$$34. \int \frac{\sqrt{a+bu} du}{u} = 2\sqrt{a+bu+a} \int \frac{du}{u\sqrt{a+bu}}$$

$$35. \int \frac{\sqrt{a+bu} du}{u^m} = -\frac{(a+bu)^{3/2}}{a(m-1)u^{m-1}} - \frac{b(2m-5)}{2a(m-1)} \int \frac{\sqrt{a+bu} du}{u^{m-1}}$$

Forms containing $\sqrt{u^2 \pm a^2}$

In this group of formulas we may replace

$$\begin{aligned} \ln(u + \sqrt{u^2 + a^2}) &\text{ by } \sinh^{-1} \frac{u}{a}, \\ \ln(u + \sqrt{u^2 - a^2}) &\text{ by } \cosh^{-1} \frac{u}{a}, \\ \ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) &\text{ by } \sinh^{-1} \frac{a}{u}. \end{aligned}$$

36. $\int (u^2 \pm a^2)^{1/2} du = \frac{u}{2} \sqrt{u^2 \pm a^2} \pm \frac{a^2}{2} \ln(u + \sqrt{u^2 \pm a^2}) + C.$
37. $\int (u^2 \pm a^2)^{n/2} du = \frac{u(u^2 \pm a^2)^{n/2}}{m+1} \pm \frac{na^2}{n+1} \int (u^2 \pm a^2)^{n/2-1} du, (n \neq -1)$
38. $\int u(u^2 \pm a^2)^{n/2} du = \frac{(u^2 \pm a^2)^{n/2+1}}{n+2} + C. \quad (n \neq -2)$
39. $\int u^m(u^2 \pm a^2)^{n/2} du = \frac{u^{m-1}(u^2 \pm a^2)^{n/2+1}}{n+m+1} - \frac{\pm a^2(m-1)}{n+m+1} \int u^{m-2}(u^2 \pm a^2)^{n/2} du.$
40. $\int \frac{du}{(u^2 \pm a^2)^{1/2}} = \ln(u + \sqrt{u^2 \pm a^2}) + C.$
41. $\int \frac{du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{+a^2 \sqrt{u^2 \pm a^2}} + C.$
42. $\int \frac{u du}{(u^2 \pm a^2)^2} = \frac{(u^2 \pm a^2)^{1-n/2} + C}{2-n}.$
43. $\int \frac{u^2 du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{2} \sqrt{u^2 \pm a^2} - \frac{\pm a^2}{2} \ln(u + \sqrt{u^2 \pm a^2}) + C.$
44. $\int \frac{u^2 du}{(u^2 \pm a^2)^{3/2}} = \frac{u}{\sqrt{u^2 \pm a^2}} + \ln(u + \sqrt{u^2 \pm a^2}) + C.$
45. $\int \frac{u^m du}{(u^2 \pm a^2)^{3/2}} = \frac{u^{m-1}}{(m-n-1)(u^2 \pm a^2)^{n-1}} - \frac{\pm a^2(m+1)}{m-n+1} \int \frac{m-2 du}{(u^2 \pm a^2)^{n/2}}$
46. $\frac{u^m du}{(u^2 \pm a^2)^2} = \frac{u^{m+1}}{\pm a^2(n-2)(u^2 \pm a^2)^{n/2-1}} - \frac{m-n+3}{\pm a^2(n-2)} \int \frac{u^m du}{(u^2 \pm a^2)^{n/2-1}}$
47. $\int \frac{du}{u(u^2 + a^2)^{1/2}} = -\frac{1}{a} \ln\left(\frac{a + \sqrt{u^2 + a^2}}{u}\right) + C.$

$$48. \int \frac{du}{u(u^2 - a^2)^{1/2}} = \frac{1}{a} \operatorname{arc} \sec \frac{u}{a} + C.$$

$$49. \frac{du}{u^2(u^2 \pm a^2)^{1/2}} = - \frac{\sqrt{u^2 \pm a^2}}{\pm a^2 u} + C.$$

$$50. \int \frac{du}{u^3(u^2 + a^2)^{1/2}} = - \frac{\sqrt{u^2 + a^2}}{2a^2 u^2} + \frac{1}{2a^3} \ln \left(\frac{a + \sqrt{u^2 + a^2}}{u} \right) + C.$$

$$51. \int \frac{du}{u^3(u^2 - a^2)^{1/2}} = \frac{\sqrt{u^2 - a^2}}{2a^2 u^2} + \frac{1}{2a^3} \operatorname{arc} \sec \frac{u}{a} + C.$$

$$52. \int \frac{du}{u^m(u^2 \pm a^2)^{n/2}} = - \frac{1}{\pm a^2(m-1)u^{m-1}(u^2 \pm a^2)^{n/2-1}} \\ - \frac{m+n-3}{\pm a^2(m-1)} \int \frac{du}{u^{m-2}(u^2 \pm a^2)^{n/2}}$$

$$53. \frac{du}{u^m(u^2 \pm a^2)^{n/2}} = \frac{1}{\pm a^2(n-2)u^{m-1}(u^2 \pm a^2)^{n/2-1}} \\ + \frac{m+n-3}{\pm a^2(n-2)} \int \frac{du}{u^m(u^2 \pm a^2)^{n/2-1}}$$

$$54. \int \frac{(u^2 + a^2)du}{u} = \sqrt{u^2 + a^2} - a \ln \left(\frac{a + \sqrt{u^2 + a^2}}{u} \right) + C.$$

$$55. \int \frac{(u^2 - a^2)^{1/2} du}{u} = \sqrt{u^2 - a^2} - a \operatorname{arc} \sec \frac{u}{a} + C.$$

$$56. \int \frac{(u^2 \pm a^2)^{1/2} du}{u^2} = - \frac{\sqrt{u^2 \pm a^2}}{u} + \ln \sqrt{u^2 \pm a^2} + C.$$

$$57. \int \frac{(u^2 \pm a^2)^{n/2} du}{u^m} = - \frac{(u^2 \pm a^2)^{n/2+1}}{\pm a^2(m-1)u^{m-1}} \\ - \frac{m-n-3}{\pm a^2(m-1)} \int \frac{(u^2 \pm a^2)^{n/2} du}{u^{m-2}}$$

$$58. \int \frac{(u^2 \pm a^2)^{n/2} du}{u^m} = \frac{(u^2 \pm a^2)^{n/2}}{(n-m+1)u^{m-1}} + \frac{\pm a^2 n}{n-m+1} \int \frac{(u^2 \pm a^2)^{n/2-1} du}{u^m}$$

Forms containing $\sqrt{a^2 - u^2}$

$$59. \int (a^2 - u^2)^{1/2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C.$$

$$60. \int (a^2 - u^2)^{n/2} du = \frac{u(a^2 - u^2)^{n/2}}{n+1} + \frac{a^2 n}{n+1} \int (a^2 - u^2)^{n/2-1} du. \quad (n \neq -1)$$

$$61. \int u^m (a^2 - u^2)^{n/2} du = -\frac{u(a^2 - u^2)^{n/2+1}}{n+2} + C. \quad (n \neq -2)$$

$$62. \int u^m (a^2 - u^2)^{n/2} du = -\frac{u^{m+1}(a^2 - u^2)^{n/2+1}}{n+m+1} \\ + \frac{a^2(m-1)}{n+m+1} \int u^{m-2}(a^2 - u^2)^{n/2} du.$$

$$63. \int \frac{du}{(a^2 - u^2)^{1/2}} = \arcsin \frac{u}{a} + C.$$

$$64. \int \frac{du}{(a^2 - u^2)^{3/2}} = \frac{u}{a^2 \sqrt{a^2 - u^2}} + C.$$

$$65. \int \frac{u du}{(a^2 - u^2)^{n/2}} = \frac{(a^2 - u^2)^{1-n/2}}{n-2} + C.$$

$$66. \int \frac{u^2 du}{(a^2 - u^2)^{1/2}} = -\frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C.$$

$$67. \int \frac{u^2 du}{(a^2 - u^2)^{3/2}} = \frac{u}{\sqrt{a^2 - u^2}} - \arcsin \frac{u}{a} + C.$$

$$68. \int \frac{u^m du}{(a^2 - u^2)^{n/2}} = -\frac{u^{m-1}}{(m-n+1)(a^2 - u^2)^{n/2-1}} + \frac{a^2(m-1)}{m-n+1} \int \frac{u^{m-2} du}{(a^2 - u^2)^{n/2}}$$

$$69. \int \frac{u^m du}{(a^2 - u^2)^{n/2}} = \frac{u^{m+1}}{a^2(n-2)(a^2 - u^2)^{n/2-1}} - \frac{m-n+3}{a^2(n-2)} \int \frac{u^m du}{(a^2 - u^2)^{n/2-1}}$$

$$70. \int \frac{du}{u(a^2 - u^2)^{1/2}} = -\frac{1}{a} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right) + C = -\frac{1}{a} \operatorname{arccosh} \frac{a}{u} + C.$$

$$71. \int \frac{du}{u^2(a^2 - u^2)^{1/2}} = -\frac{\sqrt{a^2 - u^2}}{a^2 u} + C.$$

$$72. \int \frac{du}{u^3(a^2 - u^2)^{1/2}} = -\frac{\sqrt{a^2 - u^2}}{2a^2 u^2} - \frac{1}{2a^3} \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right) + C$$

$$= -\frac{\sqrt{a^2 - u^2}}{2a^2u^2} - \frac{1}{2a^3} \cosh^{-1} \frac{a}{u} + C.$$

$$73. \int \frac{du}{u^m(a^2 - u^2)^{n/2}} = -\frac{1}{a^2(m-1)u^{m-1}(a^2 - u^2)^{n/2-1}} + \frac{m+n-3}{a^2(m-1)} \int \frac{du}{u^{m-2}(a^2 - u^2)^{n/2}}$$

$$74. \int \frac{du}{u^n(a^2 - u^2)^{n/2}} = \frac{1}{a^2(n-2)u^{n-1}(a^2 - u^2)^{n/2-1}} + \frac{m+n-3}{a^2(n-2)} \int \frac{du}{u^n(a^2 - u^2)^{n/2-1}}$$

$$75. \int \frac{(a^2 - u^2)^{1/2} du}{u} = \sqrt{a^2 - u^2} - a \ln \left(\frac{a + \sqrt{a^2 - u^2}}{u} \right) + C$$

$$= \sqrt{a^2 - u^2} - a \cosh^{-1} \frac{a}{u} + C.$$

$$76. \int \frac{(a^2 - u^2)^{1/2} du}{u^2} = -\frac{\sqrt{a^2 - u^2}}{u} - \arcsin \frac{u}{a} + C.$$

$$77. \int \frac{(a^2 - u^2)^{n/2} du}{u^m} = -\frac{(a^2 - u^2)^{n/2+1}}{a^2(m-1)u^{m-1}} + \frac{m-n-3}{a^2(m-1)} \int \frac{(a^2 - u^2)^{n/2} du}{u^{m-2}}$$

$$78. \int \frac{(a^2 - u^2)^{n/2} du}{u^m} = \frac{(a^2 - u^2)^{n/2}}{(n-m+1)u^{m-1}} + \frac{a^2n}{n-m+1} \int \frac{(a^2 - u^2)^{n/2} du}{u^m}$$

Forms containing $\sqrt{2au \pm u^2}$

The Binomial Reduction Formulas 96-104 may be applied by writing

$$\sqrt{2au \pm u^2} = u^{1/2} (2a \pm u)^{1/2}$$

$$79. \int \sqrt{2au - u^2} du = \frac{u-a}{2} \sqrt{2au - u^2} + \frac{a^2}{2} \arcsin \left(1 - \frac{u}{a} \right) + C.$$

$$80. \int u \sqrt{2au - u^2} du = -\frac{3a^2 + au - 2u^3}{6} \sqrt{2au - u^2} + \frac{a^3}{2} \arcsin \left(1 - \frac{u}{a} \right) + C.$$

$$81. \int u^m \sqrt{2au - u^2} du = -\frac{u^{m-1}(2au - u^2)^{3/2}}{m+2}$$

$$+ \frac{a(2m+1)}{m+2} \int u^{m-1} \sqrt{2au - u^2} du.$$

$$82. \int \frac{\sqrt{2au - u^2} du}{u} = \sqrt{2au - u^2} + a \arccos \left(1 - \frac{u}{a} \right) + C.$$

$$83. \int \frac{\sqrt{2au - u^2} du}{u^2} = -\frac{2\sqrt{2au - u^2}}{u} - \arccos \left(1 - \frac{u}{a} \right) + C.$$

$$84. \int \frac{\sqrt{2au - u^2} du}{u^3} = -\frac{(2au - u^2)^{3/2}}{3au^3} + C.$$

$$85. \int \frac{\sqrt{2au - u^2} du}{u^m} = -\frac{(2au - u^2)^{3/2}}{a(2m-3)u^m} + \frac{m-3}{a(2m-3)} \int \frac{\sqrt{2au - u^2} du}{u^{m-1}}$$

$$86. \int \frac{du}{\sqrt{2au - u^2}} = \arccos \left(1 - \frac{u}{a} \right) + C.$$

$$87. \int \frac{du}{\sqrt{2au + u^2}} = \ln(u + a + \sqrt{2au + u^2}) + C.$$

$$88. \int F(u, \sqrt{2au + u^2}) du = \int f(z - a, \sqrt{z^2 - a^2}) dz, \text{ where } z = u + a.$$

$$89. \int \frac{u du}{\sqrt{2au - u^2}} = -\sqrt{2au - u^2} + a \arccos \left(1 - \frac{u}{a} \right) + C.$$

$$90. \int \frac{u^2 du}{\sqrt{2au - u^2}} = -\frac{(u+3a)\sqrt{2au - u^2}}{2} + \frac{3a^2}{2} \arccos \left(1 - \frac{u}{a} \right) + C.$$

$$91. \int \frac{u^m du}{\sqrt{2au - u^2}} = -\frac{u^{m-1}\sqrt{2au - u^2}}{m} + \frac{a(2m-1)}{m} \int \frac{u^{m-1} du}{\sqrt{2au - u^2}}$$

$$92. \int \frac{du}{u\sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{au} + C.$$

$$93. \int \frac{du}{u^m \sqrt{2au - u^2}} = -\frac{\sqrt{2au - u^2}}{a(2m-1)u^m} + \frac{m-1}{a(2m-1)} \int \frac{du}{u^{m-1} \sqrt{2au - u^2}}$$

$$94. \int \frac{du}{(2au - u^2)^{3/2}} = \frac{u-a}{a^2 \sqrt{2au - u^2}} + C.$$

$$95. \int \frac{u du}{(2au - u^2)^{3/2}} = \frac{u}{a\sqrt{2au - u^2}} + C.$$

Binomial Reduction Formulas

- $$96. \int u^m(a + bu^q)^p du = \frac{u^{m-q+1}(a + bu^q)^{p+1}}{b(pq + m + 1)} - \frac{a(m - q + 1)}{b(pq + m + 1)} \int u^{m-q}(a + bu^q)^p du.$$
- $$97. \int u^m(a + bu^q)^p du = \frac{u^{m+1}(a + bu^q)^p}{pq + m + 1} + \frac{apq}{pq + m + 1} \int u^m(a + bu^q)^{p-1} du.$$
- $$98. \int \frac{du}{u^m(a + bu^q)^p} = -\frac{1}{a(m-1)u^{m-1}(a + bu^q)^{p-1}} - \frac{b(m - q + pq - 1)}{a(m-1)} \int \frac{du}{u^{m-q}(a + bu^q)^p}$$
- $$99. \int \frac{du}{u^m(a + bu^q)^p} = \frac{1}{aq(p-1)u^{m-1}(a + bu^q)^{p-1}} + \frac{m - q + pq - 1}{aq(p-1)} \int \frac{du}{u^m(a + bu^q)^{p-1}}$$
- $$100. \int \frac{d}{u(a + bu^q)} = \frac{1}{aq} \ln \left(\frac{u^q}{a + bu^q} \right) + C.$$
- $$101. \int \frac{(a + bu^q) du}{u^m} = -\frac{(a + bu^q)^{p+1}}{a(m-1)u^{m-1}} - \frac{b(m - q - pq - 1)}{a(m-1)} \int \frac{(a + bu^q)^p du}{u^{m-q}}$$
- $$102. \int \frac{(a + bu^q)^p du}{u^m} = \frac{(a + bu^q)^p}{(pq - m + 1)u^{m-1}} + \frac{apq}{pq - m + 1} \int \frac{(a + bu^q)^{p-1} du}{u^m}$$
- $$103. \int \frac{u^m du}{(a + bu^q)^p} = \frac{u^{m-q+1}}{b(m - pq + 1)(a + bu^q)^{p-1}} - \frac{a(m - q + 1)}{b(m - pq + 1)} \int \frac{u^{m-q} du}{(a + bu^q)^p}$$
- $$104. \int \frac{u^m du}{(a + bu^q)^p} = \frac{u^{m+1}}{aq(p-1)(a + bu^q)^{p-1}}$$

$$- \frac{m + q - pq + 1}{aq(p - 1)} \int \frac{u^m du}{(a + bu^q)^{p-1}}$$

Forms containing $a + bu \pm cu^2$ ($c > 0$)

The expression $a + bu + cu^2$ may be reduced to a binomial by writing

$$u = z - \frac{b}{2c}, k = \frac{b^2 - 4ac}{4c^2}$$

$$\text{Then} \quad a + bu + cu^2 = c(z^2 - k).$$

The expression $a + bu - cu^2$ may be reduced to a binomial by writing

$$u = z + \frac{b}{2c}, k = \frac{b^2 + 4ac}{4c^2}$$

$$\text{The} \quad a + bu - cu^2 = c(k - z^2).$$

$$105. \int \frac{du}{a + bu + cu^2} = \frac{2}{\sqrt{4ac - b^2}} \arctan \left(\frac{2cu + b}{\sqrt{4ac - b^2}} \right) + C, \text{ when } b^2 < 4ac.$$

$$106. \int \frac{du}{a + bu + cu^2} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \left(\frac{2cu + b - \sqrt{b^2 - 4ac}}{2cu + b + \sqrt{b^2 - 4ac}} \right) + C, \text{ when } b^2 > 4ac.$$

$$107. \int \frac{du}{a + bu - cu^2} = \frac{1}{\sqrt{b^2 + 4ac}} \ln \left(\frac{\sqrt{b^2 + 4ac} + 2cu - b}{\sqrt{b^2 + 4ac} - 2cu + b} \right) + C.$$

$$108. \int \frac{(Mu + N) du}{a + bu \pm cu^2} = \frac{\pm M}{2c} \ln(a + bu \pm cu^2) \\ + \left(N \pm \frac{bM}{2c} \right) \int \frac{du}{a + bu \pm cu^2}$$

$$109. \int \sqrt{a + bu + cu^2} du = \frac{2cu + b}{4c} \sqrt{a + bu + cu^2} \\ - \frac{b^2 - 4ac}{8c^{3/2}} \ln(2cu + b + 2\sqrt{c} \sqrt{a + bu + cu^2}) + C.$$

$$110. \int \sqrt{a + bu - cu^2} du = \frac{2cu - b}{4c} \sqrt{a + bu - cu^2} \\ + \frac{b^2 + 4ac}{8c^{3/2}} \arcsin \left(\frac{2cu - b}{\sqrt{b^2 + 4ac}} \right) + C.$$

$$111. \int \frac{du}{\sqrt{a + bu + cu^2}} = \frac{1}{\sqrt{c}} \ln (2cu + b + 2\sqrt{c} \sqrt{a + bu + cu^2}) + C.$$

$$112. \int \frac{du}{\sqrt{a + bu - cu^2}} = \frac{1}{\sqrt{c}} \arcsin \left(\frac{2cu - b}{\sqrt{b^2 + 4ac}} \right) + C.$$

$$113. \int \frac{u \, du}{\sqrt{a + bu + cu^2}} = \frac{\sqrt{a + bu + cu^2}}{c} - \frac{b}{2c^{3/2}} \ln (2cu + b + 2\sqrt{c} \sqrt{a + bu + cu^2}) + C.$$

$$114. \int \frac{u \, du}{\sqrt{a + bu - cu^2}} = -\frac{\sqrt{a + bu - cu^2}}{c} + \frac{b}{2c^{3/2}} \arcsin \left(\frac{2cu - b}{\sqrt{b^2 + 4ac}} \right) + C.$$

Other Algebraic Forms

$$115. \int \sqrt{\frac{a+u}{b+u}} \, du = \sqrt{(a+u)(b+u)} + (a-b) \log. (\sqrt{a+u} + \sqrt{b+u}) + C.$$

$$116. \int \sqrt{\frac{a-u}{b-u}} \, du = \sqrt{(a-u)(b+u)} + (a+b) \arcsin \sqrt{\frac{u+b}{a+b}} + C.$$

$$117. \int \sqrt{\frac{a+u}{b-u}} \, du = \sqrt{(a+u)(b-u)} - (a+b) \arcsin \sqrt{\frac{b-u}{a+b}} + C.$$

$$118. \int \sqrt{\frac{1+u}{1-u}} \, du = \sqrt{1-u^2} + \arcsin u + C.$$

$$119. \int \frac{du}{\sqrt{(u-a)(b-u)}} = 2 \arcsin \sqrt{\frac{u-a}{b-a}} + C.$$

Exponential and Logarithmic Forms

$$120. \int e^{au} du = \frac{e^{au}}{a} + C.$$

$$121. \int b^{au} du = \frac{b^{au}}{a \ln b} + C.$$

$$122. \int ue^{au} du = \frac{e^{au}}{a^2} (au - 1) + C.$$

$$123. \int u^n e^{au} du = \frac{u^n e^{au}}{a} - \frac{n}{a} \int u^{n-1} e^{au} du$$

$$124. \int u^n b^{au} du = \frac{u^n b^{au}}{a \ln b} - \frac{n}{a \ln b} \int u^{n-1} b^{au} du + C.$$

$$125. \int \frac{b^{au} du}{u^n} = -\frac{b^{au}}{(n-1)u^{n-1}} + \frac{a \ln b}{n-1} \int \frac{b^{au} du}{u^{n-1}}$$

$$126. \int \ln u du = u \ln u - u + C.$$

$$127. \int u^n \ln u du = u^{n+1} \left[\frac{\ln u}{n+1} - \frac{1}{(n+1)^2} \right] + C.$$

$$128. \int u^m \ln^n u du = \frac{u^{m+1}}{m+1} \ln^n u - \frac{n}{m+1} \int u^m \ln^{n-1} u du.$$

$$129. \int e^{au} \ln u du = \frac{e^{au} \ln u}{a} - \frac{1}{a} \int \frac{e^{au}}{u} du.$$

$$130. \int \frac{du}{u \ln u} = \ln(\ln u) + C.$$

Trigonometric Forms

In forms involving $\tan u$, $\cot u$, $\sec u$, $\csc u$, which do not appear below, first use the relations

$$\tan u = \frac{\sin u}{\cos u}, \cot u = \frac{\cos u}{\sin u}, \sec u = \frac{1}{\cos u}, \csc u = \frac{1}{\sin u}$$

$$131. \int \sin u du = -\cos u + C.$$

132. $\int \cos u \, du = \sin u + C.$
133. $\int \tan u \, du = -\ln |\cos u| + C = \ln |\sec u| + C.$
134. $\int \cot u \, du = \ln |\sin u| + C.$
135. $\int \sec u \, du = \int \frac{du}{\cos u} = \ln |\sec u + \tan u| + C.$
 $= \ln \tan \left(\frac{u}{2} + \frac{\pi}{4} \right) + C.$
136. $\int \csc u \, du = \int \frac{du}{\sin u} = \ln |\csc u - \cot u| + C$
 $= \ln \tan \frac{u}{2} + C.$
137. $\int \sec^2 u \, du = \tan u + C.$
138. $\int \csc^2 u \, du = -\cot u + C.$
139. $\int \sec u \tan u \, du = \sec u + C.$
140. $\int \csc u \cot u \, du = -\csc u + C.$
141. $\int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + C.$
142. $\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + C.$
143. $\int \cos^n u \sin u \, du = -\frac{\cos^{n+1} u}{n+1} + C.$
144. $\int \sin^n u \cos u \, du = \frac{\sin^{n+1} u}{n+1} + C.$
145. $\int \sin mu \sin nu \, du = -\frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C.$
146. $\int \cos mu \cos nu \, du = \frac{\sin(m+n)u}{2(m+n)} + \frac{\sin(m-n)u}{2(m-n)} + C.$
147. $\int \sin mu \cos nu \, du = -\frac{\cos(m+n)u}{2(m+n)} - \frac{(\cos m - n)u}{2(m-n)} + C.$
148. $\int \frac{du}{1 + \cos a \cos u} = 2 \csc a \arctan \left(\tan \frac{1}{2} a \tan \frac{1}{2} u \right) + C.$

$$149. \int \frac{du}{\cos a + \cos u} = \csc a \ln \left(\frac{1 + \tan \frac{1}{2} a \tan \frac{1}{2} u}{1 - \tan \frac{1}{2} a \tan \frac{1}{2} u} \right) + C$$

$$(\tan^2 \frac{1}{2} u < \cot^2 \frac{1}{2} a)$$

$$= 2 \csc a \tanh^{-1} \left(\tan \frac{1}{2} a \tan \frac{1}{2} u \right) + C \quad (\tan^2 \frac{1}{2} u < \cot^2 \frac{1}{2} a)$$

$$150. \int \frac{du}{1 + \cos a \sin u} = 2 \csc a \arctan \left(\csc a \tan \frac{1}{2} u + \cot a \right) + C.$$

$$151. \int \frac{du}{\cos a + \sin u} = \csc a \ln \left(\frac{\tan a - \tan \frac{1}{2} u - \sec a}{\tan a + \tan \frac{1}{2} u + \sec a} \right) + C.$$

$$[(\cot a \tan \frac{1}{2} u + \csc a)^2 < 1]$$

$$= -2 \csc a \tanh^{-1} \left(\cot a \tan \frac{1}{2} u + \csc a \right) + C$$

$$152. \int \frac{du}{a^2 \cos^2 u + b^2 \sin^2 u} = \frac{1}{ab} \arctan \left(\frac{b \tan u}{a} \right) + C.$$

$$153. \int e^{au} \sin nu \, du = \frac{e^{au} (a \sin nu - n \cos nu)}{a^2 + n^2} + C.$$

$$154. \int e^{au} \cos nu \, du = \frac{e^{au} (n \sin nu + a \cos nu)}{a^2 + n^2} + C.$$

$$155. \int u \sin u \, du = \sin u - u \cos u + C.$$

$$156. \int u \cos u \, du = \cos u + u \sin u + C.$$

Trigonometric Reduction Formulas

$$157. \int \sin^n u \, du = -\frac{\sin^{n-1} u \cos u}{n} + \frac{n-1}{n} \int \sin^{n-2} u \, du.$$

$$158. \int \cos^n u \, du = \frac{\cos^{n-1} u \sin u}{n} + \frac{n-1}{n} \int \cos^{n-2} u \, du.$$

$$159. \int \frac{du}{\sin^n u} = -\frac{\cos u}{(n-1)\sin^{n-1}u} + \frac{n-2}{n-1} \int \frac{du}{\sin^{n-2}u}$$

$$160. \int \frac{du}{\cos^n u} = \frac{\sin u}{(n-1)\cos^{n-1}u} + \frac{n-2}{n-1} \int \frac{du}{\cos^{n-2}u}$$

$$161. \int \cos^m u \sin^n u \, du = \frac{\cos^{m-1}u \sin^{n+1}u}{m+n} + \frac{m-1}{m+n} \int \cos^{m-2}u \sin^n u \, du.$$

$$162. \int \cos^m u \sin^n u \, du = -\frac{\sin^{n-1}u \cos^{m+1}u}{m+n} + \frac{n-1}{m+n} \int \cos^m u \sin^{n-2}u \, du.$$

$$163. \int \frac{du}{\cos^m u \sin^n u} = \frac{1}{(m-1)\sin^{n-1}u \cos^{m-1}u} \\ + \frac{m+n-2}{m-1} \int \frac{du}{\cos^{m-2}u \sin^n u}$$

$$164. \int \frac{du}{\cos^m u \sin^n u} = -\frac{1}{(n-1)\sin^{n-1}u \cos^{m-1}u} \\ + \frac{m+n-2}{n-1} \int \frac{du}{\cos^m u \sin^{n-2}u}$$

$$165. \int \frac{\cos^m u \, du}{\sin^n u} = -\frac{\cos^{m+1}u}{(n-1)\sin^{n-1}u} - \frac{m-n+2}{n-1} \int \frac{\cos^m u \, du}{\sin^{n-2}u}$$

$$166. \int \frac{\cos^m u \, du}{\sin^n u} = \frac{\cos^{m-1}u}{(m-n)\sin^{n-1}u} + \frac{m-1}{m-n} \int \frac{\cos^{m-2}u \, du}{\sin^n u}$$

$$167. \int \frac{\sin^n u \, du}{\cos^m u} = \frac{\sin^{n+1}u}{(m-1)\cos^{m-1}u} - \frac{n-m+2}{m-1} \int \frac{\sin^n u \, du}{\cos^{m-2}u}$$

$$168. \int \frac{\sin^n u \, du}{\cos^m u} = -\frac{\sin^{n-1}u}{(n-m)\cos^{m-1}u} + \frac{n-1}{n-m} \int \frac{\sin^{n-2}u \, du}{\cos^m u}$$

$$169. \int \tan^n u \, du = \frac{\tan^{n-1}u}{n-1} - \int \tan^{n-2}u \, du.$$

$$170. \int \operatorname{ctn}^n u \, du = \frac{\operatorname{ctn}^{n-1}u}{n-1} - \int \operatorname{ctn}^{n-2}u \, du.$$

$$171. \int e^{au} \cos^n u \, du = \frac{e^{au} \cos^{n-1}u (a \cos u + n \sin u)}{a^2 + n^2} \\ + \frac{n(n-1)}{a^2 + n^2} \int e^{au} \cos^{n-2}u \, du.$$

$$172. \int e^{au} \sin^n u \, du = \frac{e^{au} \sin^{n-1} u (a \sin u - n \cos u)}{a^2 + n^2} + \frac{n(n-1)}{a^2 + n^2} \int e^{au} \sin^{n-2} u \, du.$$

$$173. \int u^m \cos au \, du = \frac{u^{m-1}}{a^2} (au \sin au + m \cos au) - \frac{m(m-1)}{a^2} \int u^{m-2} \cos au \, du.$$

$$174. \int u^m \sin au \, du = \frac{u^{m-1}}{a^2} (m \sin au - au \cos au) - \frac{m(m-1)}{a^2} \int u^{m-2} \sin au \, du.$$

Inverse Trigonometric Functions

$$175. \int \arcsin u \, du = u \arcsin u + \sqrt{1-u^2} + C.$$

$$176. \int \arccos u \, du = u \arccos u - \sqrt{1-u^2} + C.$$

$$177. \int \arctan u \, du = u \arctan u - \ln \sqrt{1+u^2} + C.$$

$$178. \int \operatorname{arccot} u \, du = u \operatorname{arccot} u + \ln \sqrt{1+u^2} + C.$$

$$179. \int \operatorname{arcsec} u \, du = u \operatorname{arcsec} u - \ln(u + \sqrt{u^2-1}) + C. \\ = u \operatorname{arcsec} u - \cosh^{-1} u + C.$$

$$180. \int \operatorname{arccsc} u \, du = u \operatorname{arccsc} u + \ln(u + \sqrt{u^2-1}) + C \\ = u \operatorname{arccsc} u + \cosh^{-1} u + C.$$

Hyperbolic Functions

$$181. \int \sinh u \, du = \cosh u + C.$$

182. $\int \cosh u \, du = \sinh u + C.$
183. $\int \tanh u \, du = \ln \cosh u + C.$
184. $\int \operatorname{ctnh} u \, du = \ln \sinh u + C.$
185. $\int \operatorname{sech} u \, du = \operatorname{arc tan} (\sinh u) + C$
186. $\int \operatorname{csch} u \, du = \ln \tanh \frac{1}{2} u + C.$
187. $\int \operatorname{sech}^2 u \, du = \tanh u + C.$
188. $\int \operatorname{csch}^2 u \, du = -\operatorname{ctnh} u + C.$
189. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C.$
190. $\int \operatorname{csch} u \operatorname{ctnh} u \, du = -\operatorname{csch} u + C.$
191. $\int \sinh^2 u \, du = \frac{1}{4} \sinh 2u - \frac{1}{2} u + C.$
192. $\int \cosh^2 u \, du = \frac{1}{4} \sinh 2u + \frac{1}{2} u + C.$
193. $\int \tanh^2 u \, du = u - \tanh u + C.$
194. $\int \operatorname{ctnh}^2 u \, du = u - \operatorname{ctnh} u + C.$
195. $\int u \sinh u \, du = u \cosh u - \sinh u + C.$
196. $\int u \cosh u \, du = u \sinh u - \cosh u + C.$
197. $\int \sinh^{-1} u \, du = u \sinh^{-1} u - \sqrt{1 + u^2} + C.$
198. $\int \cosh^{-1} u \, du = u \cosh^{-1} u - \sqrt{u^2 - 1} + C.$
199. $\int \tanh^{-1} u \, du = u \tanh^{-1} u + \frac{1}{2} \ln (1 - u^2) + C.$
200. $\int \operatorname{ctnh}^{-1} u \, du = u \operatorname{ctnh}^{-1} u + \frac{1}{2} \ln (1 - u^2) + C.$
201. $\int \operatorname{sech}^{-1} u \, du = u \operatorname{sech}^{-1} u + \operatorname{gd} (\tanh^{-1} u) + C$
 $= u \operatorname{sech}^{-1} u + \operatorname{arc sin} u + C.$
202. $\int \operatorname{csch}^{-1} u \, du = u \operatorname{csch}^{-1} u + \sinh^{-1} u + C.$

$$203. \int \sinh mu \sinh nu \, du = \frac{\sinh (m+n)u}{2(m+n)} - \frac{\sinh (m-n)u}{2(m-n)} + C. \quad (m \gtrless n)$$

$$204. \int \cosh mu \cosh nu \, du = \frac{\sinh (m+n)u}{2(m+n)} + \frac{\sinh (m-n)u}{2(m-n)} + C. \quad (m \gtrless n)$$

$$205. \int \sinh mu \cosh nu \, du = \frac{\cosh (m+n)u}{2(m+n)} + \frac{\cosh (m-n)u}{2(m-n)} + C. \quad (m \gtrless n)$$

$$206. \int \frac{du}{\cos a + \cosh u} = 2 \operatorname{csch} a \tanh^{-1} \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a \right) + C.$$

$$207. \int \frac{du}{\cos a + \cosh u} = 2 \operatorname{csc} a \operatorname{arc} \tan \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a \right) + C.$$

$$208. \int \frac{du}{1 + \cos a \cosh u} = 2 \operatorname{csc} a \tanh^{-1} \left(\tanh \frac{1}{2} u \tan \frac{1}{2} a \right) + C.$$

$$\left(\tanh^2 \frac{1}{2} u < \cot^2 \frac{1}{2} a \right)$$

$$209. \int e^{au} \sinh nu \, du = \frac{e^{au}(a \sinh nu - n \cosh nu)}{a^2 - n^2} + C.$$

$$210. \int e^{au} \cosh nu \, du = \frac{e^{au}(a \cosh nu - n \sinh nu)}{a^2 - n^2} + C.$$